A Fuzzy Multilevel Mathematical Programming

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Keyworkks:

Multilevel programming, Fuzzy programming

Abstract:

This paper gives an overview on a technique for solving single objective or multiobjective mathematical programming with fuzzy parameters in the objective functions, under the presentation of multi- decision makers in different hierarchy level (decentralized decision making situation); where the higher level in the hierarchy can only influence aprather than dictate the choices of the lower level.

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1- Introduction

The use of mathematical programming in decision making has a long been restricted to problems with either single overriding objective, or a variety of objective dealt with striving toward all of them simultaneously. All objectives are assumed to be those of single decision maker (DM) who has control overall decision variables [6], [8].

The single- level can be visualized as a centralized decision making system where the hgihest level in the hierarchy (headquarters, top management, or government) has enough power to dictate the decision and have them executed by lower levels (subordinates, subunite or individuals).

A more realistic formulation, however, would recognize the role of lower level as a part of the decision proccess rather than confining them within the execution of the orders of their superiors. A multilevel linear programming (MLP), a nested optimization problem, emerged as the appropriate model to formulate such decentralized decisio making where the higher level in the hierarchy take into account the reaction of the lower level decision makers when the impact of their decision is too important to be ignored [7].

This paper also, presents the concept that use the fuzzy set theory [1], [2], [4], [5] to solve such problem in order that the obtained results from using mathematical programming techniques could be used with a higher degree of confidence.

2- Problem Formulation

A fuzzy multilevel linear programming (FMLP) is similar to standard fuzzy linear programming (FLP), except that the constraint region is modified by including a defined linear objective function; it is a nested optimization model involuing multiple n problems the first is the upper one and the $n^{\underline{th}}$ is the lower one [7]. The general form of (FMLP) can be defined as

EMLP (1):

$$\max_{X_1} f_1(x_1, x_2, \ldots, x_n) = \tilde{c}_1^{(1)} x_1 + \tilde{c}_2^{(1)} x_2 + \ldots + \tilde{c}_n^{(1)} x_n$$

where x_2 solves

$$\max_{X_2} f_2(x_1, x_2, \ldots, x_n) = \tilde{c}_1^{(2)} x_1 + \tilde{c}_2^{(2)} x_2 + \ldots + \tilde{c}_n^{(2)} x_n$$

where x_{n-1} solves

$$\max_{X_{n-1}} f_{n-1}(x_1, x_2, \ldots, x_n) = \tilde{c}_1^{(n-1)} x_1 + \tilde{c}_2^{(n-1)} x_2 + \ldots + \tilde{c}_n^{(n-1)} x_n$$

where x_n solves

$$\max_{X_n} f_n(x_1, x_2, \ldots, x_n) = \tilde{c}_1^{(n)} x_1 + \tilde{c}_2^{(n)} x_2 + \ldots + \tilde{c}_n^{(n)} x_n$$

subject to

$G_i(x_1, x_2, \ldots, x_n) \leq b,$

$x_1, x_2, \ldots, x_n \geq 0$

where $\tilde{c}_i^{(1)}, \tilde{c}_i^{(2)}, \ldots, \tilde{c}_i^{(n)}$, $i = 1,2,\ldots,n$ are a vector of fuzzy parameters and b is a constant vector; G_i $(x_1, x_2, \ldots, x_n)$ $i = 1, 2, \ldots, n$ are constraints functions; $(x_1, x_2, \ldots, x_n)$ are vectors of decision variables for each level; $f_1, f_2, \ldots, f_n$ are the objective fun[illegible] [illegible]nding to each decision maker (level) respectively (the up[illegible] [illegible]ective is often referred to as the objective of FMLP problem, while the lower objective is considered as just a constraints). Formulation FMLP visualizes an organizational hierarchy in which n decision makes have to *improve* their strategies organizational hierarchy in which n decision makes have to improve their strategies from jointly dependent set

$$S = \{(x_i) | A_i x_i \leq b, x_i \geq 0, i = 1, 2, \ldots, n\}.$$

The first decision maker who has control over x_1 make his decision first, then fixing $x_1 = x_1^o$ before the second DM select x_2, and then the resulting problem for the second DM is:

FMLP (2):

$$\max_{X_2} f_2(x_1^0, x_2, \ldots, x_n) = \tilde{c}_1^{(2)} x_1^0 + \tilde{c}_2^{(2)} x_2 + \ldots + \tilde{c}_n^{(2)} x_n$$

where x_{n-1} solves

$$\max_{X_{n-1}} f_{n-1}(x_1^0, x_2, \ldots, x_n) = \tilde{c}_1^{(n-1)} x_1^0 + \tilde{c}_2^{(n-1)} x_2 + \ldots + \tilde{c}_n^{(n-1)} x_n$$

where x_n solves

$$\max_{X_n} f_n(x_1^0, x_2, \ldots, x_n) = \tilde{c}_1^{(n)} x_1^0 + \tilde{c}_2^{(n)} x_2 + \ldots + \tilde{c}_n^{(n)} x_n$$

subject to

$G_i(x_1^0, x_2, \ldots, x_n) \leq b,$

$x_1^0, x_2, \ldots, x_n \geq 0$

and then fixing $x_2 = x_2^0$ bfore the third DM select x_3 and so on to the $n^{\underline{th}}$ decision maker.

Definltion 2.1: (memborshlp)

A real fuzzy number $\tilde{P}$ is a convex continuous fuzzy subset of the real line whose membership function $\mu_{\tilde{P}}(P_1)$ is defined by

1- A continuous mapping from R to the closed interval [0,1]

2- $\mu_{\tilde{P}}(P_i) = 0$ for all $p \in (-\infty, P_1]$

3- *Stricyly increasing on* $[P_1, P_2]$

4- $\mu_{\tilde{P}}(P_i) = 1$ for all $p \in [P_2, P_3]$

5- Strictly decreasing on $[P_3, P_4]$

6- $\mu_{\tilde{P}}(P_i) = 0$ for all $p \in [P_1, \infty)$

figure (2.1) illustrate the graph of the possible shape of membership function of a fuzzy nuber $\tilde{P}$

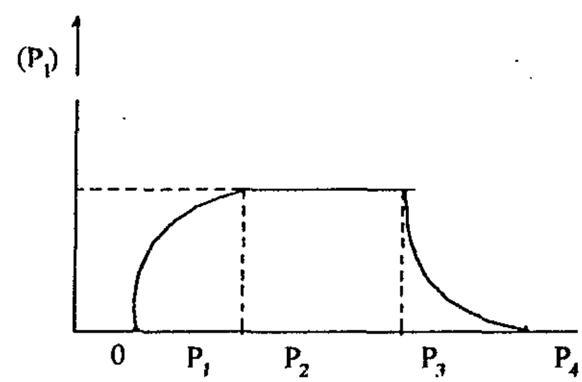


Figure (2.1) membersiungp function of fuzzy number $\tilde{P}$

Beflnltion (2.2): (α - cut)

The α- level set (α- cut) of the fuzzy numbers $\tilde{c}_i$ is defined as the ordinary set $L_\mu(\tilde{c})$ for which the degree of their membership function exceeds the level α where

$L_\mu(\tilde{c}) = \{c | \mu_{\tilde{c}_i}(C_i) \geq \alpha\}$

3- Deterministic Form for (FMLP) Problem

In this neclion the determinstic form for the fuzzy mltilevel linear mltilevel linear programming problem (α - FMLP) is detmined and then a technique for determining its corresponding solutions also prroposed as:

For the first level (first DM).

α- **FMLP (1):**

$$\max_{X_1} f_1(x_1, x_2, \ldots, x_n) = c_1^{(1)} x_1 + c_2^{(1)} x_2 + \ldots + c_n^{(1)} x_n$$

where $(x_2, x_3, \ldots, x_n)$ solves

$$\max_{X_2} f_2(x_1, x_2, \ldots, x_n) = c_1^{(2)} \qquad x_2 + \ldots + c_n^{(2)} x_n$$

where x_{n-1} solves

$$\max_{X_{n-1}} f_{n-1}(x_1, x_2, \ldots, x_n) = c_1^{(n-1)} x_1 + c_2^{(n-1)} x_2 + \ldots + c_n^{(n-1)} x_n$$

where x_n solves

$$\max_{X_n} f_n(x_1, x_2, \ldots, x_n) = c_1^{(n)} x_1 + c_2^{(n)} x_2 + \ldots + c_n^{(n)} x_n$$

subject to

$G_i(x_1, x_2, \ldots, x_n) \leq b,$

$s_1 \leq c_i^{(1)} \leq T_i$

$x_1, x_2, \ldots, x_n \geq 0$

For the second level (second DM)

α- **EMLP (2):**

$$\max_{X_2} f_2(x_1^0, x_2, \ldots, x_n) = c_1^{(2)} x_1^0 + c_2^{(2)} x_2 + \ldots + c_n^{(2)} x_n$$

where x_{n-1} solves

$$\max_{X_{n-1}} f_{n-1}(x_1^0, x_2, \ldots, x_n) = c_1^{(n-1)} x_1^0 + c_2^{(n-1)} x_2 + \ldots + c_n^{(n-1)} x_n$$

where x_n solves

$$\max_{X_n} f_n(x_1^0, x_2, \ldots, x_n) = c_1^{(n)} x_1^0 + c_2^{(n)} x_2 + \ldots + c_n^{(n)} x_n$$

subject to

$G_i (x_1^0, x_2, \ldots, x_n) \leq b,$

$S_i \leq C_i^{(2)} \leq T_i$

$x_1^0, x_2, \ldots, x_n \geq 0$

and so on to the $n^{\underline{th}}$ decision maker (level).

Then the Kuhm- Tucker conditions for the above multilevel mathematical programming model at the point $(\overline{X}_1, \overline{X}_2, \ldots, \overline{X}_n)$ and the coefficient $(\overline{C}_1, \overline{C}_2, \ldots, \overline{C}_n)$ can be determined as:

$$\frac{\partial L_1(\bar{x}, \bar{c}, \lambda, \phi, \psi, \mu)}{\partial x_1} = 0 \quad ,$$

$$\frac{\partial L_1(\bar{x}, \bar{x}, \lambda, \phi, \psi, \mu)}{\partial c_1^{(1)}} = 0 \quad ,$$

$$\frac{\partial L_1(\bar{x}, \bar{c}, \lambda, \phi, \psi, \mu)}{\partial x_2} = 0 \quad ,$$

$$\frac{\partial L_1(\bar{x}, \bar{c}, \lambda, \phi, \psi, \mu)}{\partial c_i^{(2)}} = 0 \quad ,$$

$$\frac{\partial L_n(\bar{x}, \bar{c}, \lambda, \phi, \psi, \mu)}{\partial x_n} = 0 \quad ,$$

$$\frac{\partial L_n(\bar{x}, \bar{c}, \lambda, \phi, \psi, \mu)}{\partial c_i^{(n)}} = 0 \quad ,$$

$$\left(\sum_{i=1}^{n} G_i(X_1) - b_1\right) \leq 0$$

$$\lambda_1^T \left(\sum_{i=1}^{n} G_i(X_1) - b_1\right) = 0$$

$$\phi_i^T \left(S_1^{(1)} - c_1^{(1)}\right) = 0 \quad , \quad i = 1, 2, \ldots\ldots, n \ ,$$

$$\psi_i^T \left(c_1^{(1)} - T_1^{(1)}\right) = 0 \quad , \quad i = 1, 2, \ldots\ldots, n \ ,$$

$$\phi_i^T \left(S_1^{(2)} - c_1^{(2)}\right) = 0 \quad , \quad i = 1, 2, \ldots\ldots, n \ ,$$

$$\psi_i^T \left(c_1^{(2)} - T_1^{(2)}\right) = 0 \quad , \quad i = 1, 2, \ldots\ldots, n \ ,$$

$$\phi_i^T \left(S_i^{(n)} - c_i^{(n)}\right) = 0 \quad , \quad i = 1, 2, \ldots\ldots, n \ ,$$

$$\psi_i^T \left(c_i^{(n)} - T_i^{(n)}\right) = 0 \quad , \quad i = 1, 2, \ldots\ldots, n \ ,$$

$$\mu_i^T x_i = 0 \ x_i \geq 0 \ , i = 1, 2, \ldots\ldots, n \ ,$$

where

$L_i(x, c, \lambda, \phi, \psi, \mu) =$

$$f_i(x_1, x_2, \ldots, x_n) + \sum_{i=1}^{n} \lambda_i^T \left(\sum_{i=1}^{n} G_i(X_i) - b_1\right) + \sum_{j1}^{n} \sum_{i=1}^{n} \phi_1^{(j)}$$

$$^T\left(S_1^{(j)} - C_i^{(j)}\right) + \sum_{j=1}^{n} \sum_{i=1}^{n} \psi_i^{(j)} \ ^T\left(c_i^{(j)} - T_i^{(j)}\right) + \mu_i^T x_i$$

4- An illustrative Example

Consider the following fuzzy Multilevel programming problem, the example formulate in bilevel programming form to clarify the idea; so it takes the form

FMLP:

$$\max_X f_1(x, y) = c_1^{(1)} x + c_2^{(1)} y \ ,$$

where y solves

$$\max_{y} f_2\ (x, y) = c_1^{(2)} x + c_2^{(2)} y \ ,$$

subject to

$x + y \geq 4$,

$x + 2y \geq 6$,

$$S_1^1 \leq C_1^{(1)} \leq T_1^1 \ ,$$

$$S_1^1 \leq C_1^{(1)} \leq T_1^1 \ ,$$

$$S_1^2 \leq C_1^{(2)} \leq T_1^2 \ ,$$

$$S_2^2 \leq C_2^{(2)} \leq T_2^2 \ ,$$

$x, y \geq 0$

and the corresponding lagrangian functions [3] for α- EMLP is

$L_j (x, y, c, \lambda) =$

$$\left(c_1^{(j)} x + c_2^{(j)} y\right) - \lambda_1^j \left(x + y - 4\right) - \lambda_2^j \left(x + 2y - 6\right) - \lambda_3^j \left(T_1^1 - c_1^{(1)}\right)$$
$$- \lambda_4^j \left(c_1^{(1)} - S_1^1\right) - \lambda_6^j \left(c_2^{(1)} - S_2^1\right) - \lambda_7^j \left(T_1^2 - c_1^{(2)}\right) - \lambda_8^j \left(c_1^{(2)} - S_1^2\right)$$
$$- \lambda_9^j \left(T_2^2 - c_2^{(2)}\right) - \lambda_{10}^j \left(c_2^{(2)} - S_2^2\right) - \lambda_{11}^j x - \lambda_{12}^j y,\ j = 1.2$$

Then the corresponding Kuhn- Tucker are

$$\frac{\partial L_j}{\partial x} = c_1^{(j)} - \lambda_1^j - \lambda_2^j - \lambda_{11}^j = 0, \quad \frac{\partial L_j}{\partial y} = c_1^{(j)} - \lambda_1^j - 2\lambda_2^j - \lambda_{12}^j = 0 \ , \ j = 1, 2,$$

$$\frac{\partial L_1}{\partial c_1^{(1)}} = x + \lambda_3^1 - \lambda_4^1 = 0 \quad , \quad \frac{\partial L_1}{\partial c_2^{(1)}} = y + \lambda_5^1 - \lambda_6^1 = 0$$

$$\frac{\partial L_2}{\partial c_1^{(2)}} = \lambda_7^1 - \lambda_8^2 = 0 \quad , \quad \frac{\partial L_1}{\partial c_2^{(2)}} = \lambda_9^1 - \lambda_{10}^1 = 0$$

$$\frac{\partial L_2}{\partial c_1^{(1)}} = \lambda_3^2 - \lambda_4^2 = 0 \quad , \quad \frac{\partial L_2}{\partial c_2^{(1)}} = \lambda_5^2 - \lambda_6^2 = 0$$

$\lambda_1^j \left(x + y - 4\right) = 0 \quad , \quad \lambda_2^j \left(x + 2y - 6\right) = 0 \;,\; j = 1,2 \;,$

$\lambda_3^j \left(T_1^1 - c_1^{(1)}\right) = 0 \quad , \quad \lambda_4^j \left(c_1^{(1)} - S_1^1\right) = 0 \;,\; j = 1,2 \;,$

$\lambda_5^j \left(T_2^1 - c_2^{(1)}\right) = 0 \quad , \quad \lambda_6^j \left(c_2^{(1)} - S_2^1\right) = 0 \;,\; j = 1,2 \;,$

$\lambda_7^j \left(T_1^2 - c_1^{(2)}\right) = 0 \quad , \quad \lambda_8^j \left(c_1^{(2)} - S_1^2\right) = 0 \;,\; j = 1,2 \;,$

$\lambda_9^j \left(T_2^2 - c_2^{(2)}\right) = 0 \quad , \quad \lambda_{10}^j \left(c_2^{(2)} - S_2^2\right) = 0 \;,\; j = 1,2 \;,$

$\lambda_{11}^j \; x = 0 \quad , \quad \lambda_{12}^j \; y = 0 \;,\; j = 1,2$

$\left(x + y - 4\right) \geq 0 \quad , \quad \left(x + 2y - 6\right) \geq 0 \;,$

$\left(T_1^1 - c_1^{(1)}\right) = 0 \quad , \quad \left(c_1^{(1)} - S_1^1\right) = 0 \;,$

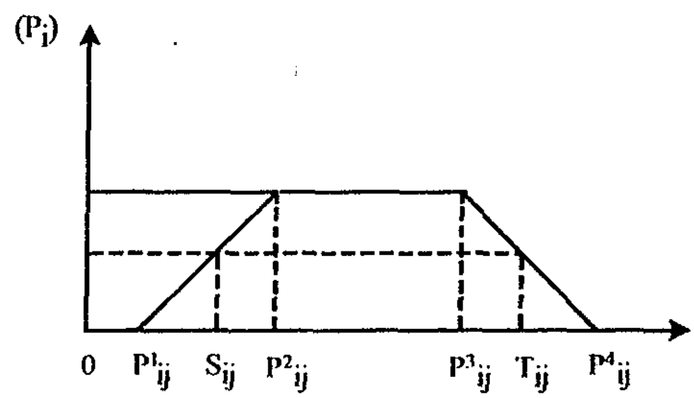
$\left(T_2^1 - c_2^{(1)}\right) = 0 \quad , \quad \left(T_2^2 - c_1^{(2)}\right) = 0 \;,$

$\left(T_1^2 - c_1^{(2)}\right) = 0 \quad , \quad \left(c_1^{(2)} - S_1^2\right) = 0 \;,$

$\left(c_2^{(2)} - S_2^2\right) = 0 \quad , \quad \left(T_2^2 - c_2^{(2)}\right) = 0 \;,$

and $x \geq 0,\ y \geq 0,\ \lambda_k^j \geq 0 \quad , \quad j = 1, 2, \ldots, 12$

To obtain $c_1^{(1)}$, $c_2^{(1)}$, $c_1^{(2)}$, and $c_2^{(2)}$ Take the cut $\alpha = 0.5$



$$\mu_{oij}\left(\bar{c}_{ij}\right) = \begin{cases} 0 & , \; -\infty < c_{ij} \le P_{i1} \; , \\ \dfrac{c_{ij} - P_{i2}}{P_{i2} - P_{i1}} & , \; P_{i2} \le c_{ij} \le P_{i2} \\ 1 & , \; P_{i2} \le c_{ij} \le P_{i3} \\ \dfrac{c_{ij} - P_{i4}}{P_{i3} - P_{i4}} & , \; P_{i3} \le c_{ij} \le P_{i4} \\ 0 & , \; P_{i4} \le c_{ij} \le \infty \end{cases}$$

c_j^i	P_{ij}^1	P_{ij}^2	P_{ij}^3	P_{ij}^4	S_{ij}	T_{ij}
c_1^1	3	5	6	8	6	7
c_2^1	2	4	6	8	5	7
c_2^2	4	6	8	10	7	9
c_2^2	3	5	6	10	6	8

Now the two system K-T (1), K-T(2) can be solve to obtain the value of the lagrangian multipliers and the solution set for the systems, where the parameters $c_1^{(1)}$, $c_2^{(1)}$, $c_1^{(2)}$, and $c_2^{(2)}$ can be obtained from the previous table.

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البرمجة الرياضية المتعددة المستويات الفازية

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ملخص البحث:

يعطى هذا البحث عرض لأسلوب حل مشاكل البرمجة الرياضية (المتعددة دوال الهدف او الوحيدة الهدف) ذات الوسائط الفازية وذلك فى حالات وجود أكثر من متخذ قرار فى تسلسل إدارى هرمى (عدم المركزية فى إتخاذ القرار).

ويعتبر هذا النوع من المشاكل هو الشائع فى العديد من المؤسسات والهيئات والحكومات حيث يكون لمتخذى القرار ذوى المستويات الأدارية الأعلى اتخاذ القرار واضعيين فى الإعتبار قرارات المعاونين لهم ذوى المتسويات الإدارية الأقل.