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A SUGGESTED SOLUTION FOR TORSION PROBLEM OF HOMOGENEOUS PRISMATIC BAR

حل مقترح لمسألة التواء على قضيب منشورى متجانس

H. M. NOUR

FACULTY OF ENGINEERING, MANSOURA UNIVERSITY, EGYPT

الملخص

نستخدم المواد غير المعدنية فى تطبيقات كثيرة كمواد هندسية. يقترح هذا البحث حلاً عددياً لمعادلة العزم لمنشور مصنوع من مادة متجانسة و متباينة الخواص خطيا. و هو مقتطع طولياً من أسطوانة مجوفة و له مقطع على شكل قطاع دائرى. لا تؤثر أى قوى خارجية على السطح الخارجى للمنشور بينما تؤثر مجموعتين من القوى الموزعة على طرفى المنشور لهما تأثير عزمى التوائى فى اتجاهين متضادين. و يتلخص الخوارزم المقترح فى الخطوات التالية:

أ- تحويل المعادلة التفاضلية الجزئية من الإحداثيات الكرتيزية الى القطبية لتناسب الشروط الحدية للمسألة.

ب- فرض حل للمعادلة التفاضلية فى شكل متسلسلة تقاربيه بواسطة معامل فيزيائى قيمته أصغر من الوحدة.

ج- استنتاج مركبات الإجهاد كمتسلسلات تقاربيه بواسطة تحليلات فورير.

تم تصميم برنامج للحاسب و بواسطته أمكن حل المسألة على منشور مصنوع من مادة لها خواص متشابهة و أخرى لها خواص متباينة لتوضيح الحل المقترح ، كما أمكن الحصول على نتائج عددية مقبولة و مناقشتها ،

ABSTRACT

Non-metallic materials are being used increasingly as engineering materials in a wide range of applications. This work introduces a numerical solution for pure torsion equation of a continuously homogeneous prismatic bar, which is cut longitudinally from a hollow cylinder, having a material with rectilinear anisotropy and a circular sector cross section. The lateral surface of the prism is free from external forces, and from all restraints. Body forces are absent, and the ends are subjected to distributed forces, which are leading to twisting moments of opposite directions. Only two stress components are different from zero, and the remaining four vanish. The problem is constructed using a small physical parameter, which characterizes the material anisotropy. A mathematical model is derived using Fourier series analysis, and the stress components are obtained as conversion expansions. Computer program is designed, and illustrative examples for orthotropic and non-orthotropic materials are introduced for showing the effectiveness of the suggested solution. Acceptable numerical results are obtained, and convergence is discussed.

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* KEY WORDS

PURE TORSION, CYLINDRICAL PRISMATIC BAR.
RECTILINEAR ANISOTROPY, STRESS COMPONENTS.

1- INTRODUCTION

In modern industry, there are many kinds of anisotropic materials which are natural like wood or synthetic like polymers, glass-fiber, or reinforced plastic. This non-metallic materials are better than metallic materials in different practical applications as aircraft construction, and medical instruments. This work introduces a numerical solution for pure torsion problem of homogeneous, anisotropic prismatic bar. The theory of generalized torsion was first worked out by Voigt, and the rigorous theory of pure torsion was developed by Saint-Venant. There are several works on the theory of pure torsion, and among these are large monograph mentioned by Lekhnitskii [1], Timoshinko [2] , and Sarkisyan [3].

Here, a pure torsion problem is stated and solved for a rod having curvilinear quadrangle cross section , while its material is linearly anisotropic .

2- STATEMENT OF THE PROBLEM

Consider the continuously homogeneous prismatic bar, which is cut from a hollow cylinder having inner radius (a) and outer radius (b), the prism's cross section is a circular sector having angle (α). The origin of coordinates is at the center of the edge cross section which lies at XY plane, and Z axis coincides with the axis of the hollow cylinder. The cylinder's material has rectilinear anisotropy and the planes of the cross sections are planes of elastic symmetry. The forces being distributed over the ends are reduced at either of them to a twisting moment M_t . Four stress components out of six are zeros : $\sigma_x = \sigma_y = \sigma_z = \tau_{xy} = 0$, and the others are related to the stress function $\psi(x, y)$ as

$$\tau_{xz}(x,y) = \frac{\partial\psi(x,y)}{\partial y} \quad \text{, and} \qquad \tau_{yz}(x,y) = -\frac{\partial\psi(x,y)}{\partial x} \tag{1}$$

where σ_x, σ_y and σ_z are the normal stress components and τ_{xy}, τ_{xz} and τ_{yz} are the tangent stress components. The stress function $\psi(x, y)$ satisfies the second-order partial differential equation

$$a_{44}\frac{\partial^2\psi}{\partial x^2} - 2a_{45}\frac{\partial^2\psi}{\partial x\partial y} + a_{55}\frac{\partial^2\psi}{\partial y^2} = -2\vartheta \tag{2}$$

where a_{44}, a_{45} and a_{55} are coefficients of elasticity and ϑ is the angle of twist per unit length [1] . The torsion rigidity C_t is given by:

$$C_t = \frac{2}{9}\iint \psi(x,y).dx.dy \tag{3}$$

The stress function $\psi(x,y)$ vanishes on the contour of the cross section

$$\psi(x,y)|_{contour} = 0 \tag{4}$$

3- A SUGGESTED SOLUTION TECHNIQUE

Equation (2) is transformed from Cartesian coordinates into polar coordinates [2] where $\psi(x,y) = \phi(r,\theta)$ as follows:

$$T[\phi(r,\theta)] + \delta.S[\phi(r,\theta)] = -A_0 \tag{5}$$

where T[ ] and S[ ] are two differential operators,

$$T[\;] = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} \text{ , and}$$

$$S[\;] = a_1(\theta)\left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right) - 2a_2(\theta)\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial \theta}\right) \text{ ,}$$

$$a_1(\theta) = \cos(2\theta) - K_1\sin(2\theta)\text{, and} \qquad a_2(\theta) = \sin(2\theta) + K_1\cos(2\theta)$$

where δ is a small physical parameter which is always less than unity [3,4]

$$\delta = \frac{a_{44} - a_{55}}{a_{44} + a_{55}} \text{ ,} \quad 0 \le \delta < 1 \text{ ,} \quad K_1 = \frac{2a_{45}}{a_{44} - a_{55}} \text{ , and} \quad A_0 = \frac{49}{a_{44} + a_{55}}$$

and the stress components, and torsion rigidity are given as:

$$\tau_{xz}(r,\theta) = \sin(\theta)\frac{\partial\phi(r,\theta)}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial\phi}{\partial\theta} \text{ .} \tag{6-a}$$

$$\tau_{yz}(r,\theta) = -\cos(\theta)\frac{\partial\phi(r,\theta)}{\partial r} + \frac{\sin(\theta)}{r}\frac{\partial\phi(r,\theta)}{\partial r} \text{ , and} \tag{6-b}$$

$$C_t = \frac{2}{9}\iint_{domain} \phi(r,\theta).rdr.d\theta \tag{6-c}$$

Solution of equation (5) is assumed in the form [5,6]

$$\phi(r,\theta) = \eta(\phi_0(r,\theta) + \delta.\phi_1(r,\theta) + \delta^2\,\phi_2(r,\theta) + \ldots\ldots\ldots) = \eta\sum_{j=0,1,2,3,\ldots}^{\infty}\delta^j.\phi_j(r,\theta) \tag{7}$$

Substituting from equation (7) into equation (5) and equating coefficients of δ^j, (j=0,1,2,...) we obtain the differential equations

$$T[\phi_0] = \frac{-A_0}{\eta} = f_0 \text{ ,} \tag{8}$$

$$T[\phi_1] = -S[\phi_0] = f_1(r,\theta)\text{ ,} \tag{9}$$

$$T[\phi_2] = -S[\phi_1] = f_2(r,\theta) \tag{10}$$

in general, $\quad T[\phi_j] = -S[\phi_{j-1}] = f_j(r.\theta)$ and $f_0 = \frac{-A_0}{\eta}$

The boundary conditions in equation (4) will be

$$\phi_j(a,\theta) = \phi_j(b,\theta) = 0 \text{ , and } \qquad \phi_j(r,0) = \phi_j(r,\alpha) = 0$$

Substituting from equation (7) into equation (6), the stress components and torsion rigidity will be

$$\tau_{xz}(r,\theta) = \eta(\tau_{0xz}(r,\theta) + \delta.\tau_{1xz}(r,\theta) + \delta^2.\tau_{2xz}(r,\theta) + \ldots\ldots\ldots) = \eta \sum_{j=0,1,2,3\ldots}^{\infty} \delta^j.\tau_{jxz}(r,\theta) \tag{11}$$

$$\tau_{yz}(r,\theta) = \eta(\tau_{0yz}(r,\theta) + \delta.\tau_{1yz}(r,\theta) + \delta^2.\tau_{2yz}(r,\theta) + \ldots\ldots\ldots) = \eta \sum_{j=0,1,2,3\ldots}^{\infty} \delta^j.\tau_{jyz}(r,\theta) \tag{12}$$

and $$C_t = \eta(C_{0t} + \delta.C_{1t} + \delta^2.C_{2t} + \ldots\ldots) = \eta \sum_{j=0,1,2,3\ldots}^{\infty} \delta^j.C_{jt} \tag{13}$$

where the first approximation in equations (11), (12), and (13) are

$$\tau_{0yz}(r,\theta) = -\cos(\theta)\frac{\partial\phi_0(r,\theta)}{\partial r} + \frac{\sin(\theta)}{r}\frac{\partial\phi_0(r,\theta)}{\partial r}, \tag{14-a}$$

$$\tau_{0xz}(r,\theta) = \sin(\theta)\frac{\partial\phi_0(r,\theta)}{\partial r} + \frac{\cos(\theta)}{r}\frac{\partial\phi_0(r,\theta)}{\partial\theta}, \text{ and} \tag{14-b}$$

$$C_{0t} = \frac{2}{\vartheta}\int_0^\alpha\int_a^b \phi_0(r,\theta).rdrd\theta \tag{14-c}$$

Now equation (8) is solved for the first approximation $\phi_0(r,\theta)$ using Fourier series (sine half-range expansion) [7] for both f_0 and $\phi_0(r,\theta)$ as:

$$f_0 = \sum_{k=1,3,5\ldots}^{\infty} b_k.\sin(\lambda_k\theta), \qquad \phi_0(r,\theta) = \sum_{k=1,3,5\ldots}^{\infty} R_k(r).\sin(\lambda_k\theta)$$

$$\text{where} \quad b_k = \frac{1}{k}, \qquad \eta = \frac{-16\vartheta}{\pi(a_{44}+a_{55})}, \quad \text{and} \quad \lambda_k = \frac{\pi k}{\alpha} \tag{15}$$

Substituting from equations (15) into equation (8) and obtaining the ordinary differential equation

$$\frac{d^2R_k(r)}{dr^2} + \frac{1}{r}\frac{dR_k(r)}{dr} - \frac{\lambda_k^2}{r^2}R_k(r) = b_k \tag{16}$$

which has a complete solution in the form [8]

$$R_k(r) = C_k.r^{\lambda_k} + D_k.r^{-\lambda_k} + V_k(r) \tag{17}$$

and $\phi_0(r,\theta)$ in equation (15) will be in the form

$$\phi_0(r,\theta) = \sum_{k=1,3,5\ldots}^{\infty}(C_k.r^{\lambda_k} + D_k.r^{-\lambda_k} + V_k(r)).\sin(\lambda_k\theta) \tag{18-a}$$

$$\text{where} \quad C_k = b^{-\lambda_k}\frac{V_k(a) - V_k(b).C^{-\lambda_k}}{C^{-\lambda_k} - C^{\lambda_k}}, \qquad D_k = b^{\lambda_k}\frac{V_k(b).C^{\lambda_k} - V_k(a)}{C^{-\lambda_k} - C^{\lambda_k}},$$

$$C = \frac{a}{b}, \qquad V_k(r) = V_K.r^2, \text{ and} \qquad V_k = \frac{1}{k(4-\lambda_k^2)}, \qquad \lambda_k \neq \pm 2$$

Substituting from equation (18) into equations (14)

$$\tau_{0yz}(r,\theta) = \frac{1}{2}\sum_{k=1,3,5...}^{\infty} -\sin((\lambda_k - 1)\theta).L_1 + \sin((\lambda_k + 1)\theta).L_2 \ , \tag{18-b}$$

$$\tau_{0xz}(r,\theta) = \frac{1}{2}\sum_{k=1,3,5...}^{\infty} \cos((\lambda_k - 1)\theta).L_1 + \cos((\lambda_k + 1)\theta).L_2 \ , \tag{18-c}$$

$$C_{0t} = 4b^2 \sum_{k=1,3,5...}^{\infty} \frac{1}{\lambda_k}(C_k.b^{\lambda_k}\frac{1-C^{\lambda_k+2}}{\lambda_k+2} + D_k.b^{-\lambda_k}\frac{1-C^{2-\lambda_k}}{2-\lambda_k} + V_k.b^2\frac{1-C^4}{4}) \tag{18-d}$$

where $L_1 = 2\lambda_k.C_k.r^{\lambda_k-1} + (\lambda_k+2)V_k.r$ and

$L_2 = 2\lambda_k.D_k.r^{-\lambda_k-1} + (\lambda_k-2)V_k.r$

Repeating this procedure to solve equation (9) for $\phi_1(r,\theta)$,and equation (10) for $\phi_2(r,\theta)$, then obtaining the second and the third approximations in equations (11), (12), and (13) as

$$\phi_1(r,\theta) = \sum_{n=2,4,6...}^{\infty}(C_n.r^{\lambda_n} + D_n.r^{-\lambda_n} + V_n(r)).\sin(\lambda_n\theta) \tag{19-a}$$

where $C_n = b^{-\lambda_n}\dfrac{V_n(a) - V_n(b).C^{-\lambda_n}}{C^{-\lambda_n} - C^{\lambda_n}}$, $D_n = b^{\lambda_n}\dfrac{V_n(b).C^{\lambda_n} - V_n(a)}{C^{-\lambda_n} - C^{\lambda_n}}$,

$\lambda_n = \dfrac{\pi n}{\alpha}$, $V_n(r) = \sum_{k=1,3,5...}^{\infty} U_{n,k}.r^{\lambda_k} + W_{n,k}.r^{-\lambda_k} + Q_{n,k}.r^2$

$U_{n,k} = E(n,k)\dfrac{4C_k.\lambda_k(\lambda_k-1)}{\lambda_n^2 - \lambda_k^2}$, $W_{n,k} = F(n,k)\dfrac{4D_k.\lambda_k(\lambda_k+1)}{\lambda_n^2 - \lambda_k^2}$, $\lambda_n \neq \lambda_k$

$Q_{n,k} = \dfrac{V_k.\lambda_k}{4-\lambda_n^2}\left[2(F(n,k) - E(n,k)) - \lambda_k(F(n,k) + E(n,k))\right]$

$$F(n,k) = \frac{1}{2}\left(\frac{-\sin(\alpha_1) - K_1.\cos(\alpha_1)}{\alpha_1} + \frac{\sin(\alpha_2) - K_1.\cos(\alpha_2)}{\alpha_2} + \frac{2K_1.\pi n}{\alpha_1.\alpha_2}\right)$$

$$E(n,k) = \frac{1}{2}\left(\frac{\sin(\alpha_3) + K_1.\cos(\alpha_3)}{\alpha_3} + \frac{-\sin(\alpha_4) + K_1.\cos(\alpha_4)}{\alpha_4} - \frac{2K_1.\pi n}{\alpha_3.\alpha_4}\right)$$

$\alpha_1 = \pi(n+k) + 2\alpha$, $\alpha_2 = \pi(n-k) - 2\alpha$

$\alpha_3 = \pi(n-k) + 2\alpha$, $\alpha_4 = \pi(n+k) - 2\alpha$

$$\tau_{1yz} = \frac{1}{2}\sum_{n=2,4,6...}^{\infty} -\sin((\lambda_n - 1)\theta).M_1 + \sin((\lambda_n + 1)\theta).M_2 \tag{19-b}$$

$$\tau_{1xz} = \frac{1}{2}\sum_{n=2,4,6...}^{\infty} \cos((\lambda_n - 1)\theta).M_1 + \cos((\lambda_n + 1)\theta).M_2 \tag{19-c}$$

$$M_1 = 2\lambda_n.C_n.r^{\lambda_n-1} + \sum_{k=1,3,5...}^{\infty}(\lambda_n+\lambda_k)U_{n,k}.r^{\lambda_k-1} + (\lambda_n-\lambda_k)W_{n,k}.r^{-\lambda_k-1} + (\lambda_n+2)Q_{n,k}.r$$

$$M_2 = 2\lambda_n.D_n.r^{-\lambda_n-1} + \sum_{k=1,3,5...}^{\infty}(\lambda_n-\lambda_k)U_{n,k}.r^{\lambda_k-1} + (\lambda_n+\lambda_k)W_{n,k}.r^{-\lambda_k-1} + (\lambda_n-2)Q_{n,k}.r$$

$$C_{1t} = 0 \tag{19-d}$$

$$\phi_2(r,\theta) = \sum_{m}^{\infty}(C_m.r^{\lambda_m} + D_m.r^{-\lambda_m} + V_m(r)).\sin(\lambda_m\theta) \tag{20-a}$$

$$C_m = b^{-\lambda_m}\frac{V_m(a) - V_m(b).C^{-\lambda_m}}{C^{-\lambda_m} - C^{\lambda_m}}, \qquad D_m = b^{\lambda_m}\frac{V_m(b).C^{\lambda_m} - V_m(a)}{C^{-\lambda_m} - C^{\lambda_m}}, \qquad \lambda_m = \frac{\pi m}{\alpha}$$

$$V_m(r) = \sum_{n=2,4,6...}^{\infty} B_1.r^{\lambda_n} + B_2.r^{-\lambda_n} + \sum_{k=1,3,5...}^{\infty} B_3.r^{\lambda_k} + B_4.r^{-\lambda_k} + B_5.r^2$$

$$B_1 = \frac{-4C_n.\lambda_n.(\lambda_n - 1).E(m,n)}{\lambda_n^2 - \lambda_m^2}, \qquad B_2 = \frac{-4D_n.\lambda_n.(\lambda_n + 1).F(m,n)}{\lambda_n^2 - \lambda_m^2}, \qquad \lambda_n \neq \lambda_m$$

$$B_3 = \frac{-2U_{n,k}.(\lambda_n + \lambda_k)(\lambda_k - 1)E(m,n)}{\lambda_k^2 - \lambda_m^2} + \frac{2U_{n,k}.(\lambda_n + \lambda_k)(\lambda_k - 1)F(m,n)}{\lambda_k^2 - \lambda_m^2} + \frac{-4C_k.\lambda_k(\lambda_k - 1)H_{m,k}}{\lambda_k^2 - \lambda_m^2}$$

$$B_4 = \frac{-2W_{n,k}.(\lambda_k - \lambda_n)(\lambda_k + 1)E(m,n)}{\lambda_k^2 - \lambda_m^2} + \frac{-2W_{n,k}.(\lambda_n + \lambda_k)(\lambda_k + 1)F(m,n)}{\lambda_k^2 - \lambda_m^2} + \frac{-4D_k.\lambda_k.(\lambda_k + 1)G_{m,k}}{\lambda_k^2 - \lambda_m^2}$$

$$B_5 = \frac{-2Q_{n,k}.(\lambda_n + 2)E(m,n)}{4 - \lambda_m^2} + \frac{2Q_{n,k}.(\lambda_n - 2)F(m,n)}{4 - \lambda_m^2} + \frac{V_k\,\lambda_k(4 - \lambda_k^2)G_{m,k}}{(4 - \lambda_m^2)(2 + \lambda_k)} + \frac{-V_k.\lambda_k(4 - \lambda_k^2)H_{m,k}}{(2 - \lambda_k)(4 - \lambda_m^2)}$$

$$\lambda_k \neq \lambda_m \quad, \lambda_m \neq \pm 2$$

$$G_{m,k} = \frac{1 - K_1^2}{4}\left(\frac{-\sin(\gamma_1)}{\gamma_1} + \frac{\sin(\gamma_2)}{\gamma_2}\right) - \frac{K_1}{2}\left(\frac{\cos(\gamma_1)}{\gamma_1} + \frac{\cos(\gamma_2)}{\gamma_2} - \frac{2\pi m}{\gamma_1.\gamma_2}\right)$$

$$H_{m,k} = \frac{1 - K_1^2}{4}\left(\frac{\sin(\gamma_3)}{\gamma_3} - \frac{\sin(\gamma_4)}{\gamma_4}\right) + \frac{K_1}{2}\left(\frac{\cos(\gamma_3)}{\gamma_3} + \frac{\cos(\gamma_4)}{\gamma_4} - \frac{2\pi m}{\gamma_3.\gamma_4}\right)$$

$$\gamma_1 = \pi(m + k) + 4\alpha, \qquad \gamma_2 = \pi(m - k) - 4\alpha$$

$$\gamma_3 = \pi(m - k) + 4\alpha, \qquad \gamma_4 = \pi(m + k) - 4\alpha$$

$$\tau_{2yz} = \frac{1}{2}\sum_{m=1,2,3...}^{\infty} -\sin((\lambda_m - 1)\theta).S_1 + \sin((\lambda_m + 1)\theta).S_2 \qquad (20\text{-b})$$

$$\tau_{2xz} = \frac{1}{2}\sum_{m=1,2,3...}^{\infty} \cos((\lambda_m - 1)\theta).S_1 + \cos((\lambda_m + 1)\theta).S_2 \qquad (20\text{-c})$$

$$S_1 = 2\lambda_m.C_m.r^{\lambda_m - 1} + \sum_{n=2,4,6...}^{\infty} B_1(\lambda_m + \lambda_n)r^{\lambda_n - 1} + B_2(\lambda_m - \lambda_n)r^{-\lambda_n - 1} +$$

$$+ \sum_{k=1,3,5...}^{\infty} B_3(\lambda_m + \lambda_k)r^{\lambda_k - 1} + B_4(\lambda_m - \lambda_k)r^{-\lambda_k - 1} + B_5(\lambda_m + 2)r$$

$$S_2 = 2\lambda_m.D_m.r^{-\lambda_m - 1} + \sum_{n=2,4,6...}^{\infty} B_1(\lambda_m - \lambda_n)r^{\lambda_n - 1} + B_2(\lambda_m + \lambda_n)r^{-\lambda_n - 1} +$$

$$+ \sum_{k=1,3,5...}^{\infty} B_3(\lambda_m - \lambda_k)r^{\lambda_k - 1} + B_4(\lambda_m + \lambda_k)r^{-\lambda_k - 1} + B_5(\lambda_m - 2)r$$

$$C_{2t} = 4b^2 \sum_{m=1,3,5...}^{\infty} \frac{1}{\lambda_m}\Big(C_m.b^{\lambda_m}\frac{1 - C^{2+\lambda_m}}{2 + \lambda_m} + D_m.b^{-\lambda_m}\frac{1 - C^{2-\lambda_m}}{2 - \lambda_m} +$$

$$+ \sum_{n=2,4,6...}^{\infty} B_1.b^{\lambda_n}\frac{1 - C^{2+\lambda_n}}{2 + \lambda_n} + B_2.b^{-\lambda_n}\frac{1 - C^{2-\lambda_n}}{2 - \lambda_n} + \sum_{k=1,3,5...}^{\infty} B_3.b^{\lambda_k}\frac{1 - C^{2+\lambda_k}}{2 + \lambda_k} +$$

$$+ B_4.b^{-\lambda_k}\frac{1 - C^{2-\lambda_k}}{2 - \lambda_k} + B_5.b^2\frac{1 - C^4}{4}\Big)$$

(20-d)

ALGORITHM DEVELOPMENT

1- INPUT:

i) Coefficients of elasticity (a_{44}, a_{55}, a_{45}), and angle of twist per unit length (ϑ).

ii) Sector's dimensions (a, b, α).

iii) Coordinates (r, θ) for the point at which stress components and torsion rigidity are calculated.

2- Calculate (δ, K_1) by using formulas (5), and (η, λ_k) by using formula (15).

3- Calculate ($V_k, V_k(r), C_k, D_k$), then $\phi_0(r,\theta)$ by using equation (18-a).

4- Calculate (L_1, L_2), then τ_{0yz} by using equation (18-b), τ_{0xz} by using equation (18-c), and C_{0t} by using equation (18-d).

5- Calculate ($\alpha_1, \alpha_2, \alpha_3, \alpha_4$), (F(n,k), E(n,k)), ($U_{n,k}, W_{n,k}, Q_{n,k}$), $V_n(r)$, (λ_n, C_n, D_n), and $\phi_1(r,\theta)$ by using equation (19-a).

6- Calculate (M_1, M_2), τ_{1yz} by using equation (19-b), and τ_{1xz} by using equation (19-c).

7- Calculate ($\gamma_1, \gamma_2, \gamma_3, \gamma_4$), ($G_{m,k}, H_{m,k}$), (F(m,n), E(m,n)), ($B_1, B_2, B_3, B_4, B_5$), $V_m(r)$, (λ_m, C_m, D_m), and $\phi_2(r,\theta)$ by using equation (20-a).

8- Calculate (S_1, S_2), τ_{2yz} by using equation (20-b), τ_{2xz} by using equation (20-c), and C_{2t} by using equation (20-d).

9- OUTPUT:

i) The stress function $\phi(r,\theta)$ by using equation (7).

ii) The stress components, $\tau_{xz}(r,\theta)$ by using equation (11), and $\tau_{yz}(r,\theta)$ by using equation (12).

iii) Torsion rigidity C_t by using equation (13).

Computer program is designed in FORTRAN language, and runs on personal computer. Double precession leads to more convergence and stability than single precision. The following tables are chosen as samples of the obtained results.

4- CASE STUDIES

In the following, all numerical values are calculated in double precision form, all written results are performed to the fixed significant figures in the last two successive approximations, and at the point where $(r,\theta) = (a + \frac{b-a}{2}, \frac{\alpha}{2})$.

EXAMPLE (1):

a- Consider an orthotropic prism ($a_{45} = 0$), having sector dimensions as; b= unit length, C=0.2, $\alpha = \pi/4$. Tables (1, 2) give double precision numerical values for the first and third approximations for stress function, stress components and torsion rigidity against truncation number, as in equations (18), and (20) respectively.

Table (1)

k	$10\times\phi_0$	k	$100\times\tau_{0xz}$	k	$10\times\tau_{0yz}$	k	$100\times C_{0t}$
5	-0.184	5	-0.99	15	0.239	17	-0.7023
13	-0.1844	15	-0.993	33	0.2399	33	-0.70234
29	-0.18445	33	-0.9937	87	0.23990	81	-0.702346
63	-0.184453	79	-0.99373	89	0.239909	99	-0.7023462

In equations (19-a, b, c, d), for k=100, and n=8 the results $\phi_1 \approx 0.0$, $\tau_{1xz} = -0.383398\times10^{-2}$, $\tau_{1yz} = -0.158808\times10^{-2}$, and $C_{1t} = 0.0$ are obtained. No more accuracy is gained from n=8 to n=100

Table (2) k=100, n=100

M	$10^{4}\times\phi_2$	M	$10\times\tau_{2xz}$	M	$10\times\tau_{2yz}$	M	$10\times C_{2t}$
7	0.5	2	-0.7	2	-0.3	6	0.23
19	0.54	8	-0.74	6	-0.33	8	0.233
33	0.547	24	-0.744	31	-0.339	18	0.2338
59	0.5474	58	-0.7444	44	-0.3391	58	0.23389

Tables (1.2) show the solution's convergence and stability.

b-Consider an orthotropic prism ($a_{45} = 0$), having sector dimensions as; b=10 unit length, $\alpha = \pi/4$. Table (3) gives double precision numerical values for stress function, stress components τ_{xz}, τ_{yz}, and torsion rigidity C_t for first, second, and third approximations from equations (18), (19), and (20), for several values of C.

Table (3) k=75, n=76, m=77

	C=0.2	C=0.4	C=0.6	C=0.8
$0.1\times\phi_0$	-0.184453	-0.187506	-0.12960	-0.0389
ϕ_1	≈ 0.0	≈ 0.0	≈ 0.0	≈ 0.0
$100\times\phi_2$	0.547	0.069	-0.91892	0.6495
$10\times\tau_{0xz}$	-0.9937	-0.2792	0.0969	0.0541
τ_{1xz}	-0.038	-0.1014	-0.145	-0.0626
$0.1\times\tau_{2xz}$	-0.0452573	-0.1461705	-0.167801	0.183094
τ_{0yz}	0.2399	0.0674	-0.02340	-0.013077
$10\times\tau_{1yz}$	-0.158	-0.420	-0.601	-0.259
τ_{2yz}	-0.218213	-0.62419	-0.60896	0.93501
$0.01\times C_{0t}$	-0.7023458	-0.592681	-0.317603	-0.0608452
C_{1t}	0.0	0.0	0.0	0.0
$0.001\times C_{2t}$	0.1576299	0.138078	0.0709	-0.0165248

Table (3) shows that, whenever the hollow cylinder is thicker the convergence and stability is better.

EXAMPLE (2):

a- Consider non-orthotropic prism ($a_{45} \neq 0$), having sector dimensions as; b=unit length, C=0.2, $\alpha = \pi/6$, and $K_1 = 1.0$.

The first approximation in equation (18) are independent of K_1 and they are:

$100\times\phi_0$=-0.94755, $100\times\tau_{0XZ}$=-0.56657, $10\times\tau_{0YZ}$=0.21144, $100\times C_{0t}$=-0.264334.

Table (4) gives double precision numerical values for the second approximation for stress function, stress components, and torsion rigidity against truncation number, as in equations (19).

Table (4) $\phi_1 \approx 0.0$, $C_{1t} = 0$

N	$1000\times\tau_{1xz}$	$1000\times\tau_{1yz}$
2	-o.76	-0.20
4	-0.766	-0.205
6	-0.7663	-0.2053

Table (5) gives double precision numerical values for the thired approximation for stress function, stress components, and torsion rigidity against truncation number, as in equations (20).

Table (5) k=70, n=70

M	$100\times\phi_2$	M	$100\times\tau_{2xz}$	M	$0.1\times\tau_{2yz}$	M	$100\times C_{2t}$
5	-0.9	6	0.7	1	-0.2	3	0.9
7	-0.90	30	0.74	3	-0.25	8	0.95
11	-0.908	46	0.741	9	-0.255	14	0.952
29	-0.9087	54	0.7415	22	-0.2554	18	0.9521

b- Consider non-orthotropic prism ($a_{45}\neq 0$), having sector dimensions as; b= 10 unit length, C=0.2, $\alpha=\pi/6$. Table (6) gives double precision numerical values for stress components τ_{xz},τ_{yz} and torsion rigidity C_t for the first, second, and third approximations in equations (18), (19), and (20), for several values of K_1.

Table (6)

ϕ_0=-0.9475, $10\times\tau_{0XZ}$=-0.56657, τ_{0YZ}=0.2114, $0.01\times C_{0t}$=-0.264334

	K_1 =0.01	K_1 =0.1	K_1 =0.5	K_1 =1.0	K_1 =5.0
ϕ_1	≈ 0.0	≈ 0.0	≈ 0.0	≈ 0.0	≈ 0.0
ϕ_2	0.177691	0.11761	-0.18648	-0.65176	-7.77972
$10\times\tau_{1xz}$	-0.0281	-0.03247	-0.052	-0.076	-0.27
$10\times\tau_{2xz}$	0.1332	0.19915	0.4790	0.7889	1.60
$10\times\tau_{1yz}$	-0.0076	-0.0088	-0.0140	-0.0205	-0.0720
$10\times\tau_{2yz}$	0.4969	0.3388	-0.489	-1.809	-23.8183
$0.01\times C_{2t}$	0.16969	0.22813	0.45314	0.65464	-0.9237

Table (6) shows that, whenever K_1 is smaller, convergence is faster, otherwise more approximated terms are needed for acceptable accuracy.

5- CONCLUSIONS

A numerical method is introduced for solving the problem of pure torsion equation of a continuously homogeneous prismatic bar, which is cut longitudinally from hollow cylinder, having a material with rectilinear anisotropy and a circular sector cross section. Mathematical model is derived using Fourier series analysis, and the stress components are obtained as conversion expansions dependent on a small physical parameter, which characterizes the material anisotropy. The solution is suitable for orthotropic and non-orthotropic materials. Numerical results show that:

1- Whenever the physical parameter is smaller, the truncation error is smaller, and the number of approximating terms for acceptable results is lesser.

2- First approximation is always independent of coefficients of elasticity for orthotropic or non-orthotropic materials.

3- The series's summations for orthotropic materials are dependent on the prism's dimensions and independent of material's coefficients of elasticity.

4- Whenever dimensions are smaller the series converge faster and better, therefore a suitable length's unit is chosen for small errors.

5- Whenever the hollow cylinder is thicker the convergence and stability is better.

6- Personal computer needs few seconds to calculate the above results in double precision.

7- This mathematical model fails for solid cylinder, when sector's angle equals $\pm\frac{\pi}{2}$, and for non-orthotropic materials when sector angle equals $\frac{\pi}{4}$.

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