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INVESTIGATION AND ANALYSIS OF THE PRECISION CRITERIA IN FREE AND FIXED LEVELLING NETWORKS

استقصاء وتحليل معايير الدقه في شبكات الميزانيات الحره والثابته (المربوطه)

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الملخص

في السنوات الاخيره زاد إستخدام شبكات الميزانيات الحره في الاعمال الهندسيه المختلفه ، وقد ظهر جليا أنه عند مقارنــة نتائج الشبكـات الحره بعد اجراء عمليات الضبط الخاصه بها بنتائج شبكه مربـوطة أوثابته لها نفس الشكل ونفس خط الارصاد ومقاسه تحت نفس الظروف الطبيعيه وجد أن الشبكات الحره تعطي قيما عدديه لمعايير الدقه المعروفه أقل من القيم العدديه لمعايير الدقه للشبكات المربوطه وعليه فأن هذا البحث يهدف إلي دراسه هذه الظاهره لمعرفه أسبابها عن طريق تفسير إختلاف المفهوم الهندسي لمعايير الدقه في كل من شبكات الميزانيات الحره والمربوطه بهدف الوصول إلي معايير جديده مناسبه لوصف الدقه في الحالتين وأيضا يمكن عن طريقها إجراء المقارنه الصحيحه بين دقه الشبكات الحره والمربوطة ، وقد تم الاستعانه بشبكه ميزانيه دقيقه عمليه أجريت في مدينه المنصورة كمثال تطبيقي حيث تم ضبطها عده مرات بمتغيرات مختلفه لتلائم هذه الدراسة.

ABSTRACT

In the last few years, the use of free levelling networks in the different engineering constructions has been increased It was clear that comparing the results of the free levelling networks with the fixed levelling networks, having the same configurations and the same measurement technologies, yielded unexpected smaller numerical values of common precision criteria than those of the fixed levelling network.
The aim of this research is to study the reasons for this phenomena by interpreting and illustrating the variation in the geometrical meanings for precision criteria of both free and fixed levelling networks
New criteria suitable to describe the precisions in both cases have been obtained. Hence, it is possible to make a correct comparison between them. A practical levelling network has been used as a practical sample example using different variables in adjustment processes for this study

1. INTRODUCTION

Nowadays, the levelling networks are widely used as vertical control networks for many purposes of engineering control surveys such as, for detecting vertical crustal movements and for detecting of deformation of dams, bridges and other large structures.
The quality of these engineering control networks is described by measures of precision, reliability, economy and sensitivity. The precision of a levelling network may express the degree of propagation of random errors in the network All information concerning the precision of the adjusted parameters is contained in the variance -covariance matrix. Size of elements of the variance-covariance matrix expresses the precision of the network The best measure of precision should satisfy the following conditions. :

a- Invariance with respect to the choice of the computational base
b- Invariance with respect to the datum transformation.
c- Independent of the choice of the least squares adjustment techniques

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d- Characterize the level precision of points in the network.
e- Applicable to primary as well as secondary geodetic networks.

The main objective of this paper is to investigate and analyze the precision concept for the fixed and free levelling networks. Also, an important objective is to find a measure to describe the internal precision for fixed levelling networks. In this work, the adjustment for the practical levelling network is performed by using least square principles in three approaches :

I- Minimum constraints adjustment approach (holding one point level fixed).
II- Inner constraints adjustment approach (free from any outer constraints).
III- Over constrained adjustment approach (holding more than one point levels fixed).

2. PARAMETRIC ADJUSTMENT MODEL

The mathematical model for the least square adjustment by the method of observation equations is composed of the functional model, which is given in linear form by

$$v = A\,\hat{x} - l \tag{1}$$

and the stochastic model

$$\Sigma_l = \sigma^2_o\, P^{-1} \tag{2}$$

where v is the n-vector of residuals, A is the n by u configuration matrix, $\hat{x}$ is the u-vector of the unknown parameters, l is the n-vector of the absolute terms, P is the n by n weight matrix, σ^2_o is the apriori variance factor and finally Σ_l is the n by n variance-covariance matrix of the observed quantities. By implementing requirements of the method of least squares $v^T Pv$ = minimum, one gets to the system of normal equations

$$A^TPA\,\hat{x} - A^TPl = 0 \tag{4.a}$$

or

$$N\;\hat{x} - U = 0 \tag{4.b}$$

The system (4.b) can be solved easily if r = u and r = rank(A) = rank(N) and the best estimates for the weight coefficient matrix is

$$Q_{\hat{x}} = N^{-1} \tag{5}$$

In case of free network r < u , the matrix of normal equations N is singular and the Cayley inverse N^{-1} does not exist. Such a problem can be practically solved by using the so-called the Moore -Penrose pseudo inverse, or some other simple approaches like the inner constraints procedure [1,2]. However, detailed analysis of the free network approach is considered beyond the scope of the present paper.Then the estimated weight coefficient matrix of the estimated parameters is

$$Q_{\hat{x}} = N^{+} = (A^TPA)^{+} \tag{6}$$

The best estimates for the solution vector x and their corresponding variance covariance matrix $\Sigma_{\hat{X}}$ are obtained as

$$\hat{x} = Q_{\hat{X}}\, U \quad (7)$$

and

$$\Sigma_{\hat{X}} = \hat{\sigma}^2_o\, Q_{\hat{X}} \quad (8)$$

where $\hat{\sigma}^2_o$ is the aposteriori variance factor which is computed from

$$\hat{\sigma}^2_o = (v^T P v) / r \quad (9)$$

with $r = n - u$ (in case of fixed network)
or $r = n - u + d$ (in case of free network),

where r is the number of degrees of freedom (redundancies), n is the number of the observations, u is the number of the unknown parameters (point levels) and d is the datum defect. Finally, the adjusted values of the observations, as well as, their estimated variance -covariance matrix can be computed as

$$\hat{L} = L + v \quad (10)$$

and

$$\Sigma_{\hat{L}} = \hat{\sigma}^2_o\, Q_{\hat{L}} = \hat{\sigma}^2_o\, (A Q_{\hat{X}} A^T) \quad (11)$$

3. ANALYSIS OF THE PRECISION CRITERIA

3.1 The Common Precision Criteria

The variance-covariance matrix of the estimated point levels $\Sigma_{\hat{X}}$ is the source for all information required to measure the precision of the levelling network. The criteria that can measure the precision can be summarized as follow:

1- The standard error of the adjusted point level

$$\sigma_{\hat{H}_i} = \hat{\sigma}_o \sqrt{Q_{H_i H_i}} \quad (12)$$

2- The standard error of the adjusted level difference .

$$\sigma_{\hat{\Delta h}_i} = \hat{\sigma}_o \sqrt{(A\, Q_{\hat{X}}\, A^T)_{ii}} \quad (13)$$

3- The relative standard error between two levelled points Pi and Pj is given by [7]

$$\sigma_{H_iH_j} = \hat{\sigma}_o \sqrt{(Q_{H_iH_i} + Q_{H_jH_j} - 2\, Q_{H_iH_j})} \qquad (14)$$

4- The mean standard error of the levels of all the network points is defined as a global measure of precision [2,3,7,8] , and given by

$$\sigma_N = \sqrt{\frac{\mathrm{trace}(\Sigma_{\hat{x}})}{m}} \qquad (15)$$

in which m is the total number of the network points. This measure is known in German literature as mean standard level network error [9] .

5- Another global measure of precision is the geometric mean of the eigen values [2,7,8] , which is given by

$$\sigma_W = \sqrt[m]{\det\left(\Sigma_{\hat{x}}\right)} \qquad (16)$$

which is based on the determinant of the estimated variance-covariance matrix.

3.2 The Precision Concept in Fixed Levelling Network

In fixed levelling network, the mathematical model for the estimated level difference from the point P_i to any fixed point P_f in an arbitrary datum system is given by

$$\Delta h_i = H_i - H_f \qquad (17)$$

in which H_f is the errorless level of the fixed point P_f. By applying the error propagation concepts using the variance law [2] on Equation (17), it was found that

$$\sigma_{\hat{\Delta} h_i} = \sigma_{\hat{H}_i} \qquad (18)$$

Therefore, it can be concluded that, in fixed levelling network the standard error of any new point level $\sigma_{\hat{H}_i}$ is identical with the standard error of the level difference between it and the level of any arbitrary fixed point.

3.3 The Precision Concept in Free Levelling Network

It is well known that, the free levelling network has none fixed points (benchmarks). Consequently, all the level points are allowed to receive corrections after the performing of the adjustment process. The measure $\sigma_{\hat{H}_i}$ in free network can be obtained from the level difference between the new point level H_i and the average of all the levels of the network points

$$F_i = H_i - \frac{H_1 + H_2 + \cdots + H_m}{m}$$

By putting i = 1 this equation can be written as

$$F_1 = \left(\frac{m-1}{m}\right)H_1 - \frac{1}{m}\left(H_2 + \cdots + H_m\right) \tag{19}$$

By applying the law of covariance propagation [2], it was found that

$$Q_{F_1F_1} = \left(\frac{m-1}{m}\right)^2 Q_{H_1H_1} - 2\left(\frac{m-1}{m^2}\right) Q_{H_1H_2} \cdots\cdots - 2\left(\frac{m-1}{m^2}\right) Q_{H_1H_m}$$
$$+ \frac{1}{m^2} Q_{H_1H_2} + \cdots\cdots + 2\left(\frac{1}{m^2}\right) Q_{H_2H_m}$$
$$\cdots \quad \cdots\cdots$$
$$+ \frac{1}{m^2} Q_{H_mH_m}$$

which can be rearranged to

$$Q_{F_1F_1} = Q_{H_1H_1} - \frac{2m-1}{m^2}\left(Q_{H_1H_1} + Q_{H_1H_2} + \cdots\cdots + Q_{H_1H_m}\right)$$
$$+ \frac{1}{m^2}\left(Q_{H_1H_2} + Q_{H_2H_2} + \cdots\cdots + Q_{H_2H_m}\right)$$
$$\cdots\cdots$$
$$+ \frac{1}{m^2}\left(Q_{H_1H_m} + Q_{H_2H_m} + \cdots\cdots + Q_{H_mH_m}\right) \tag{20}$$

In the free network the sum of the elements between brackets for Equation (20) is zero [6]. Then,

$$Q_{F_1F_1} = Q_{H_1H_1} \qquad \text{or} \qquad \sigma_{F_1} = \sigma_{H_1} \tag{21}$$

From these results, it can be concluded that, in free levelling network the standard error of any new point level $\sigma_{\hat{H}i}$ is identical with the standard error of the level difference between it and the average of all the levels of the network points.

For the purpose of this study, the levelling network illustrated in Fig. 1 should be adjusted in eight variants. In the first and second variants, the levelling network was adjusted as a fixed network using the over constrained adjustment approach with three benchmarks (A, B & C) and with two benchmarks (A & B) respectively. In the third variant, the two benchmarks (B & C) were deleted and the levelling network was adjusted as a fixed network with one benchmark at (A) by using the minimum constrained approach. By deleting the last benchmark (A) for the fourth variant, the levelling network became free from any fixed point and it can be adjusted as a free levelling network using the inner constrained approach.

To study the change effect of the fixed position on the above external and internal precision criteria, the levelling network should be adjusted using the minimum constrained approach.

In the variants five, six, seven and eight, P_1, P_2, P_3 and P_4 were chosen as fixed points respectively.

All the computations were performed on a 386 PC computer. The computer programme used in this research to adjust the levelling network via the mentioned three approaches was developed by the researcher. Moreover, a subroutine for the estimation of the external, internal and global precision criteria for the fixed and free levelling network was created.

6- RESULTS AND ANALYSIS

I- Table (2) contains the adjusted levels of the network points $\hat{H}_i$, the adjusted level differences $\hat{\Delta h}_i$ and their corresponding standard errors $\sigma_{\hat{\Delta h}_i}$ for all the eight variants of the levelling network. From the obtained results, it was found that :

Table (2) The adjusted levels (m), level differences (m) and their standard errors (cm).

	variant (1)	variant (2)	variant (3)	variant (4)	variant (5)	variant (6)	variant (7)	variant (8)
	A, B & C are B.M.	A & B are B.M.	A is B.M.	Free Net	P_1 is fixed	P_2 is fixed	P_3 is fixed	P_4 is fixed
$\hat{H}_1$	7.4677	7.4666	7.4680	7.4693	10.0000	10.5529	11.9275	10.6719
$\hat{H}_2$	6.9144	6.9134	6.9151	6.9164	9.4471	10.0000	11.3745	10.1189
$\hat{H}_3$	5.5406	5.5390	5.5405	5.5418	8.0725	8.6255	10.0000	8.7444
$\hat{H}_4$	6.7958	6.7946	6.7961	6.7974	9.3281	9.8811	11.2556	10.0000
$\hat{\Delta h}_1$	0.5532	0.5532	0.5529	0.5529	0.5529	0.5529	0.5529	0.5529
$\hat{\Delta h}_2$	1.9271	1.9276	1.9275	1.9275	1.9275	1.9275	1.9275	1.9275
$\hat{\Delta h}_3$	0.6719	0.6720	0.6719	0.6719	0.6719	0.6719	0.6719	0.6719
$\hat{\Delta h}_4$	1.3739	1.3744	1.3745	1.3745	1.3745	1.3745	1.3745	1.3745
$\hat{\Delta h}_5$	0.1186	0.1188	0.1189	0.1189	0.1189	0.1189	0.1189	0.1189
$\hat{\Delta h}_6$	1.2552	1.2556	1.2556	1.2556	1.2556	1.2556	1.2556	1.2556
$\hat{\Delta h}_7$	1.7623	1.7634	1.7620	-	-	-	-	-
$\hat{\Delta h}_8$	1.8524	1.8514	-	-	-	-	-	-
$\hat{\Delta h}_9$	1.5366	-	-	-	-	-	-	-
$\sigma_{\hat{\Delta h}_1}$	0.218	0.235	0.281	0.281	0.281	0.281	0.281	0.281
$\sigma_{\hat{\Delta h}_2}$	0.256	0.293	0.335	0.335	0.335	0.335	0.335	0.335
$\sigma_{\hat{\Delta h}_3}$	0.247	0.267	0.305	0.305	0.305	0.305	0.305	0.305
$\sigma_{\hat{\Delta h}_4}$	0.249	0.285	0.325	0.325	0.325	0.325	0.325	0.325
$\sigma_{\hat{\Delta h}_5}$	0.258	0.280	0.321	0.321	0.321	0.321	0.321	0.321
$\sigma_{\hat{\Delta h}_6}$	0.270	0.299	0.338	0.338	0.338	0.338	0.338	0.338
$\sigma_{\hat{\Delta h}_7}$	0.318	0.393	0.604	-	-	-	-	-
$\sigma_{\hat{\Delta h}_8}$	0.316	0.392	-	-	-	-	-	-
$\sigma_{\hat{\Delta h}_9}$	0.329	-	-	-	-	-	-	-

1- Free levelling network adjustment using the inner constrained approach (variant 4) gave the same numerical values of the adjusted level differences and their standard errors as those obtained by a minimum constrained approach (variants 3,5,6,7&8). This means that, the adjusted level differences and their standard errors are independent of the least squares adjustment approaches. Consequently, they are invariant with respect to datum translation.
2- By decreasing the number of the benchmarks (fixed points) for the levelling network, the values of the standard errors of the level differences have been increased (variants 1, 2, &3).
3- The adjusted levels of the network points depend upon the choice of the zero - variance reference base. Therefore, they are not invariant with respect to datum translation.

II- Table (3) gives the common global precision criteria for all the eight variants for the levelling network. In addition, the sum of squares of weighted residuals [Pvv] and the aposteriori standard error of unit weight $\hat{\sigma}_o$ are also presented. From these results, it can be noted that :

1- The estimated aposteriori standard error $\hat{\sigma}_o$ and the [Pvv] for the free levelling network adjustment using the inner constraints (variant 4) were the same numerical values as those obtained by a minimum constrained approach (variant 3, 5, 6, 7& 8). This means that, the aposteriori standard error $\hat{\sigma}_o$ and the [Pvv] are invariant with respect to datum translation.
2- Similarly, the determinant of the weight coefficient matrix [$\det(Q_{\hat{X}})$] and the geometric mean of the eigen values σ_W were also invariant with respect to the datum translation.
3- By increasing the number of the benchmarks (fixed points) in the levelling network, the numerical values of the $\hat{\sigma}_o$, trace($Q_{\hat{X}}$), det($Q_{\hat{X}}$), σ_N and σ_W have been decreased (see variants 1, 2 &3), whereas [Pvv] has been significantly increased.
4- Trace($Q_{\hat{X}}$), Q_{NN} and σ_N were not invariant with respect to the datum translation. They were depend upon the choice of the minimum constraints.

Table (3) The common global precision criteria.

	variant (1)	variant (2)	variant (3)	variant (4)	variant (5)	variant (6)	variant (7)	variant (8)
	A, B & C are B.M.	A & B are B.M.	A is B.M.	Free Net	P_1 is fixed	P_2 is fixed	P_3 is fixed	P_4 is fixed
n	9	8	7	6	6	6	6	6
u	4	4	4	4	3	3	3	3
d	-	-	-	1	-	-	-	-
r	5	4	3	3	3	3	3	3
[Pvv] (cm^2)	1.6317	1.5135	1.4568	1.4568	1.4568	1.4568	1.4568	1.4568
$\hat{\sigma}_o$ (cm)	0.571	0.615	0.697	0.697	0.697	0.697	0.697	0.697
Trace ($Q_{\hat{X}}$) *	1.3477	1.8996	3.5930	0.3126	0.5854	0.5913	0.6848	0.6393
Q_{NN} *	0.3369	0.4749	0.8982	0.0782	0.1951	0.1971	0.2283	0.2131
σ_N (cm)	0.332	0.424	0.660	0.195	0.308	0.309	0.333	0.322
det ($Q_{\hat{X}}$) *	8.58E-4	1.43E-3	3.23E-3	4.29E-3	4.29E-3	4.29E-3	4.29E-3	4.29E-3
σ_W (cm)	0.098	0.120	0.166	0.113	0.113	0.113	0.113	0.113

* unitless

III- Table (4) shows the external and internal standard errors of the adjusted point levels as well as the external and internal mean level network errors for all the eight network variants. The comparison of the numerical values for both external and internal standard errors of the adjusted point levels showed that:

1- The external standard errors of the adjusted point levels $\sigma_{\hat{H}_i}$ as well as the external mean level network error σ_N were dependent of the choice of the reference datum.

2- The internal standard errors of the adjusted point levels $(\sigma_{\hat{H}_i})_{in}$ gave identical values for both free and fixed levelling network (variants 3, 4, 5, 6, 7& 8). This means that, they were independent of the reference datum. Therefore, they were invariant with respect to the datum translation.

3- By decreasing the number of the benchmarks (fixed points) for the levelling network, the numerical values of both the external and the internal standard errors of the new point levels have been increased (see variants 1, 2& 3).

Table (4) The external and internal standard errors of the adjusted levels (cm).

		variant (1)	variant (2)	variant (3)	variant (4)	variant (5)	variant (6)	variant (7)	variant (8)
		A, B & C are B.M.	A & B are B.M.	A is B.M.	Free Net	P_1 is fixed	P_2 is fixed	P_3 is fixed	P_4 is fixed
External Precision	$\sigma_{\hat{H}_1}$	0.318	0.393	0.604	0.182	0.000	0.281	0.335	0.305
	$\sigma_{\hat{H}_2}$	0.316	0.392	0.666	0.184	0.281	0.000	0.325	0.321
	$\sigma_{\hat{H}_3}$	0.329	0.458	0.691	0.212	0.335	0.325	0.000	0.338
	$\sigma_{\hat{H}_4}$	0.362	0.448	0.677	0.199	0.305	0.321	0.338	0.000
	σ_N	0.332	0.424	0.660	0.195	0.308	0.309	0.333	0.322
Internal Precision	$(\sigma_{\hat{H}_1})_{in}$	0.141	0.156	0.182	0.182	0.000	0.182	0.182	0.182
	$(\sigma_{\hat{H}_2})_{in}$	0.143	0.158	0.184	0.184	0.184	0.000	0.184	0.184
	$(\sigma_{\hat{H}_3})_{in}$	0.164	0.187	0.212	0.212	0.212	0.212	0.000	0.212
	$(\sigma_{\hat{H}_4})_{in}$	0.164	0.176	0.199	0.199	0.199	0.199	0.199	0.000
	$(\sigma_N)_{in}$	0.153	0.170	0.195	0.195	0.172	0.172	0.163	0.168

IV- Table (5) shows the relative standard errors between the adjusted levels of the network points $\sigma_{H_iH_j}$ based on Equation (14). It was found from the obtained results that, all the numerical values of $\sigma_{H_iH_j}$ for all the eight variants were identical with the standard errors of the adjusted level differences $\sigma_{\hat{\Delta}hi}$ in table (2). This means that, this measure describes not only the precision of the adjusted observations (level differences) but also the relative precision between each pairs of the network points. Therefore, the standard errors of the level differences can give a better idea for the internal precision of the levelling network.

Table (5) The relative standard errors between the levelling network points (cm)

	variant (1)	variant (2)	variant (3)	variant (4)	variant (5)	variant (6)	variant (7)	variant (8)
	A, B & C are B.M.	A & B are B.M	A is B.M.	Free Net	P_1 is fixed	P_2 is fixed	P_3 is fixed	P_4 is fixed
$\sigma_{H_1H_2}$	0.218	0.235	0.281	0.281	0.281	0.281	0.281	0.281
$\sigma_{H_2H_3}$	0.256	0.293	0.335	0.335	0.335	0.335	0.335	0.335
$\sigma_{H_3H_4}$	0.247	0.267	0.305	0.305	0.305	0.305	0.305	0.305
$\sigma_{H_4H_1}$	0.249	0.285	0.325	0.325	0.325	0.325	0.325	0.325
$\sigma_{H_1H_3}$	0.258	0.280	0.321	0.321	0.321	0.321	0.321	0.321
$\sigma_{H_2H_4}$	0.270	0.299	0.338	0.338	0.338	0.338	0.338	0.338
σ_{H_1A}	0.318	0.393	0.604	-	-	-	-	-
σ_{H_2B}	0.316	0.392	-	-	-	-	-	-
σ_{H_3C}	0.329	-	-	-	-	-	-	-

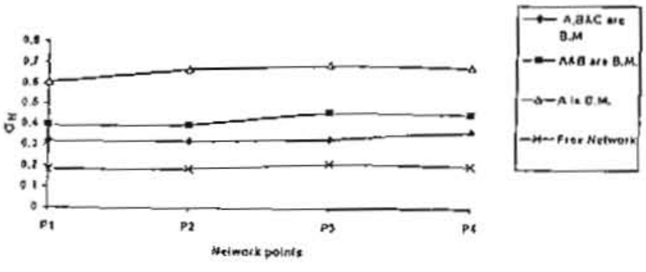

Fig. 2. The external standard errors of the adjusted levels (cm)

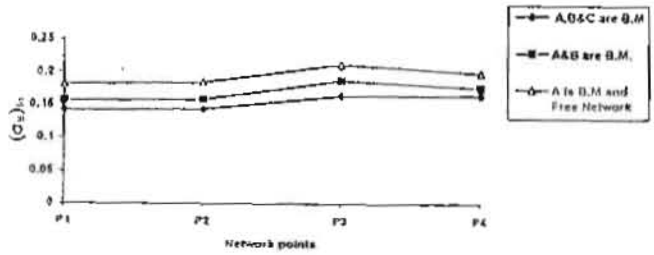

Fig. 3 The internal standard errors of the adjusted levels (cm)

7- CONCLUSIONS AND RECOMMENDATIONS

The main conclusions deduced from the analysis and discussions of this study can be summarized as follows:

1- The adjusted levels of the network points and their estimated standard errors are not invariant with respect to both the datum translation and the choice of the zero variance reference (computational) base.

2- In fixed levelling network, the standard errors of the point levels have various geometrical meaning than those for the free network. They describe the external precision for fixed network and can be named as the external standard errors of the point levels, while they describe the internal precision for the free network and can be named as the internal standard errors of the point levels.

3- Because of the external errors of the point levels are not invariant with respect to both the datum translation and the choice of the computational base, they seem to be insufficient for describing the precision for the fixed network. Therefore, the internal standard errors of the point levels for the fixed network should be calculated using Equation (20).

4- By calculating the internal standard errors of the point levels in the fixed network, the comparison between both the fixed and the free levelling network is valid.

5- The increase of the number of the fixed points or the outer constraints leads to the increase of the internal precision of the levelling network.
6- The estimated residuals and their variance-covariance matrix, the adjusted level differences and their variance -covariance matrix, the sum of squares of weighted residuals [Pvv], the determinant of the variance-covariance matrix of the point levels. the geometric mean of the eigen values and the aposteriori standard error of unit weight are invariant with respect to the datum translation and also are independent of the choice of the computational base.
7- The geometric mean of the eigen values gives a reasonable tool for measuring the global precision for the whole network.
8- The standard errors of the level differences give a better idea for the relative internal precision of the levelling network.

Finally, the precision predictions for the whole levelling network and individual stations are related to the internal mean standard level net work error and the internal standard errors of the point levels respectively. Additionally, we must remember that the adjustment procedures are based on different stochastic assumption. However, each group of precision models can be used successfully provided that the systematic errors of the system have been minimized.

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