BOUNDARY ELEMENT-CONJUGATE GRADIENT METHOD FOR ELECTROMAGNETIC SCATTERING BY CONDUCTING CYLINDERS

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الخلاصة – باستخدام طريقتى العناصر المحيطية والاتجاهات المترافقة يقدم البحث طريقة لمعالجة تشتت الموجات الكهرومغناطيسيه بواسطة أسطح أسطوانيه تامة التوصيل . وتتميز الطريقة بسهولة تنفيذها على حاسب محدود الذاكره. ومع تطبيق مبدأ الشروط المحيطية الممتدة الذى طوره واترمان يمكن كذلك معالجة مسائل التشتت عند ترددات الرنين للحيز داخل السطح الاسطوانى .

ABSTRACT- A boundary element-conjugate gradient memory economic technique is presented for the analysis of the scattering of an E or H-wave by a conducting cylinder of arbitrary cross-section. The case of internal resonance is treated using the developed technique along with Waterman's extended boundary condition.

I. INTRODUCTION

Recently, the boundary element method (BEM) has been applied to scattering and radiation problems in various engineering areas-electromagnetic wave scattering by conducting or dielectric bodies being one example [1-4] . The governing equation is the scalar Helmholtz equation from which a simple boundary integral equation is derived. Using finite element techniques to discretize the integrals and applying point collocation, the problem is reduced to solving a system of linear equations. In terms of the moment method, the BEM is equivalent to using subsectional bases and Dirac delta functions as testing functions. In the BEM, however, the shape functions corresponding to subsectional bases are taken systematically which renders the method simple and easy programmable.

By confining the analysis to the boundaries, the BEM is more memory economic than its rival finite element or finite difference methods. In this paper we consider further development of this advantage by using the conjugate gradient method (CGM) for solving the BEM equations. The CGM is one of the most efficient iterative methods for solving large systems of equations without storing any square matrices [5,6] . It has also the additional advantage of fast convergence where a reasonably accurate solution is obtained after a finite number of iterations, usually less than the order of the matrix. Therefore, the CGM offers considerable reduction in memory storage requirements without much increase in computation time.

Like conventional moment methods based on E-field or H-field integral equations, the BEM has been reported to fail to give solution to the scattering problem at frequencies corresponding to internal resonance of a cavity formed by a hollow conductor of the same shape as the original scatterer [3] . At such frequencies the coefficient matrix of the BEM equations is too ill-conditioned and nearly singular. To remedy this defect, we adopt Waterman's extended boundary condition (EBC) [7] which can be easily incorporated with the present BEM-CGM algorithm without altering the basic formulation or invoking sophisticated integral equations like, for instance, the combined source [8] or combined field [9] equations. These equations do give a unique solution at internal resonance, but require much computational effort.

II- Boundary Element Equations

The BEM formulation of electromagnetic scattering problems is given in detail in references [1-3] . We present here the main equations for convenience. We consider the two dimensional problem of scattering of an E-wave by a perfectly conducting cylinder. Let E_z and H_t be the scattered fields and $\bar{E}_z$ and $\bar{H}_t$ be the fields radiated by a line current source. By integrating the scalar Helmholtz equation and using Green's theorem we obtain the following boundary integral equation [1,2]

$$E_{zp} = \int_B \left[\bar{E}_z(r,r_p)\, H_t(r,r_p) - E_z(r,r_p)\, \bar{H}_t(r,r_p)\right] ds \qquad \ldots(1)$$

Where

$$\bar{E}_z = \frac{w\mu}{4} H_o^{(2)}(k\,|r-r_p|),$$

$$\bar{H}_t = \frac{jk}{4} H_o^{(2)}(k\,|r-r_p|)\ \hat{n}.\hat{r},$$

$$k = w\sqrt{\mu\epsilon}$$

and $\hat{n}$ is an outward unit normal to the boundary B.

Equation (1) could also be derived by applying Lorentz reciprocity theorem to the scattered and line source fields [3,10] . When the observation point p is moved to the boundary, equation(1) reduces to

$$cE_p = \int_B (\bar{E}\, H - E\, \bar{H})\, ds \qquad \ldots(2)$$

where $c = a/2\pi$ and a is the interior angle at p. The arguments as well as the subscripts t and z have been omitted for brevity and the symbol $\int$ denotes Cauchy's principal value of the integral.

Next, the contour B is discretized in the usual isoparametric finite element method and the unknown fields are approximated by, say, linear polynomials over the elements. Equation (2) becomes

$$cE_p = \sum_{j=1}^{N} \frac{B_j}{2} \left(H_j \int_{-1}^{1} \bar{E}\ \Psi_1 dx + H_{j+1} \int_{-1}^{1} \bar{E}\ \Psi_2\, dx\right)$$

$$\sum_{j=1}^{N} \frac{B_j}{2} \left(E_j \int_{-1}^{1} \bar{H}\ \Psi_1\, dx + E_{j+1} \int_{-1}^{1} \bar{H}\ \Psi_2\, dx\right) \qquad \ldots(3)$$

where $\Psi_1 = (1-x)/2$, $\Psi_2 = (1+x)/2$, and x is a normalized local coordinate along the element B_j.

Finally, equation(3) is applied at each boundary node and the boundary condition $E = -E^i$ is imposed, where E^i is the incident electric field. The resulting N equations are represented in the matrix form

$$Ah = -Be^i = b \qquad .....(4)$$

where $e^i = \text{col}\,(E_1^i, E_2^i, ..., E_N^i)$, $h = \text{col}\,(H_1, H_2, ..., H_N)$, and A and B are square matrices. Explicit expressions for the elements of A and B are given in reference [3].

In this way the scattering problem has been reduced to solving the matrix equation (4). Howver, the method obviously fails to give a unique solution at frequencies for which the homogeneous equation $Ah = 0$ has a nonzero solution. It can be shown that these are the resonance frequencies of the complementary interior problem; an H-wave inside a perfectly conducting cylinder of the same shape as the scatterer [11] . Several methods have been developed to get around this problem-a survey is given, for instance, in the reference list of [9] .

Here we adopt a simple and efficient method based on the extended boundary condition (EBC) introduced by Waterman [7, 11] . The method consists in modifying theA matrix using additional equations obtained by extending the boundary condition into some interior points. The field at these points is computed from

$$E_{zp} = \int_B (\dot{E}_z H_t - E_z \bar{H}_t)\, ds \quad p \text{ inside } B \qquad .....(5)$$

or its discretized form which is similar to equation (3). Using these additional equations results in an overdetermined system of linear equations. A least-squares solution is then obtained by minimizing the residual (error)function

$$R = \| A'h - b' \| \qquad .....(6)$$

where A' is the rectangular matrix of the overdetermined system and b' is the corresponding right-side column vector.

III- The Conjugate Gradient Method

The CGM solves the system of linear equations (4) by finding the position of the minimum of an error function similar to that given by equation (6). Thus, the iterative process for minimizing (6) is also used for solving (4). In each step of iteration a new trial vector h is computed by incrementing the last value of h by a direction vector p. The direction vector p^{i+1} is chosen to be mutually conjugate with p^i with respect to the matrix A^*A. (i.e. such that $p^{i*} A^*A\, p^{i+1} = 0$) and to be as nearly as possible in the direction of the maximum gradient of the error function at the point h^i(the direction of steepest descent). Here the star denotes ordinary complex conjugate transpose.

The conjugate gradient algorithm starts with an initial guess h^o and sets

$$r^o = b - Ah^o \text{ and } p^o = \bar{r}^o = A^* r^o \qquad .....(7)$$

Iteration then proceeds as follows

$$u^n = A p^n \qquad .....(8)$$

$$a_n = \|\bar{r}^n\| / \|u^m\| \qquad .....(9)$$

$$h^{n+1} = h^n + a_n p^n \qquad .....(10)$$

$$r^{n+1} = r^n - a_n u^n \qquad .....(11)$$

$$\bar{r}^{n+1} = A^* r^{n+1} \qquad .....(12)$$

$$b_n = \|\bar{r}^{n+1}\| / \|\bar{r}^n\| \qquad .....(13)$$

$$p^{n+1} = \bar{r}^{n+1} + b_n p^n \qquad .....(14)$$

The process is terminated when the relative change in h is less than some specified limit.

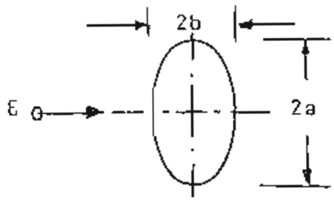
It is clear that the only operations in the conjugate gradient procedure which involves the coefficient matrix A (or A*) is matrix-vector multiplication. Since this operation can be performed quite efficiently on an element-by-element basis, memory space for the square matrix A can be dispensed with entirely. The elements of A are regenerated each time they are needed, but never stored in the high speed RAM. Alternatively, the elements may be computed at the very begining of the program and stored in backup memory (disk space) until they are called up in due time.

Both the above alternatives involve an increase in execution time, but the fast convergence of the CGM makes this increase tolerable. Thus, it is known that in the absence of rounding-off errors, the present CGM yields the exact solution after m steps, where m is the number of distinct eigenvalues of the matrix AA* [5,6]. Rounding-off errors may slow the convergence, but reasonably accurate results are often obtained after a number of iterations less than the order of the matrix. This is confirmed by the examples in the next section as well as by the numerical results of reference [12] , which applies the CGM to solve large systems of equations associated with a conventional method of moments.

IV- Numerical Examples

We have considered two problems with known analytical solutions. The first is the scattering of a plane E-wave by a perfectly conducting elliptic cylinder. The parameter computed is the forward scattering cross-section which is determined by the radiation zone scattered field in the direction of the incident wave [13]. This latter field is computed from equation (1), but with the Hankel functions replaced by their asymptotic values and the integral discretized in the usual way. The results are presented in Table 1. With 72 nodes, the difference between the analytical and numerical results is within 1.5% and is probably due to rounding off errors in both the BEM and the analytical solutions. The latter is given in terms of series of ordinary and modified Mathieu functions which are computed using Fourier series and Bessel function products series.

Table 1 gives also a comparison between the CGM and the conventional LU-decomposition. The CGM has been implemented without storing the square-coefficient-matrix. However, this CGM considerable save in memory space is achieved at the expense of much increase in the execution time, in spite of the fast convergence of the method. (Less than N/2 iteration steps have been sufficient to get close agreement with LU-decomposition results).



$a = 2\sqrt{q}\cosh\xi_o$

$b = 2\sqrt{q}\sinh\xi_o$

Table 1: Comparison between conjugate gradient method (CGM) and LU-decomposition. Analytical results are taken from [14] Number of iterations is shown in brackets.

q, ξ_o	Boundary Elements N	Normalized Cross-section σ/k LU	CGM	Analyt.	Comp. Time (Relative Units) LU	CGM
1,1	36	14.17	14.17(17)	14.65	22	111
	72	14.43	14.43(25)		46	541
1,0.4	36	9.401	9.400(18)	9.755	22	115
	72	9.602	9.602(26)		46	557
1,0.2	36	8.432	8.430(18)	8.743	22	115
	72	8.598	8.597(27)		46	572

The second example is the scattering of a plane H-wave by a conducting circular cylinder at internal resonance (ka = 3.8317 = first zero of $J_1(x)$). The formulation is basically the same as in section II with the roles of E and H interchanged and the Dirichlet's boundary condition replaced by a Neumann's condition. To predict the occurence of resonance, we could examine the singularity of the BEM matrix by computing its codition number over a range of frequencies. However, this would require computing the norm of the inverse matrix (or its largest eigenvalue) and would not be a simple task as the matrix itself is not stored in the high-speed memory. We have, therefore, adopted an alternative test based on the energy conservation at infinity. It is well known that for an incident plane wave the power scattering diagram $|F(\varphi)|^2$ satisfies the relation

$$\int_0^{2\pi} |F(\varphi)|^2 \, d\varphi = -2\pi \, \mathrm{Re}\left[F(\varphi^i) \right] \qquad .....(15)$$

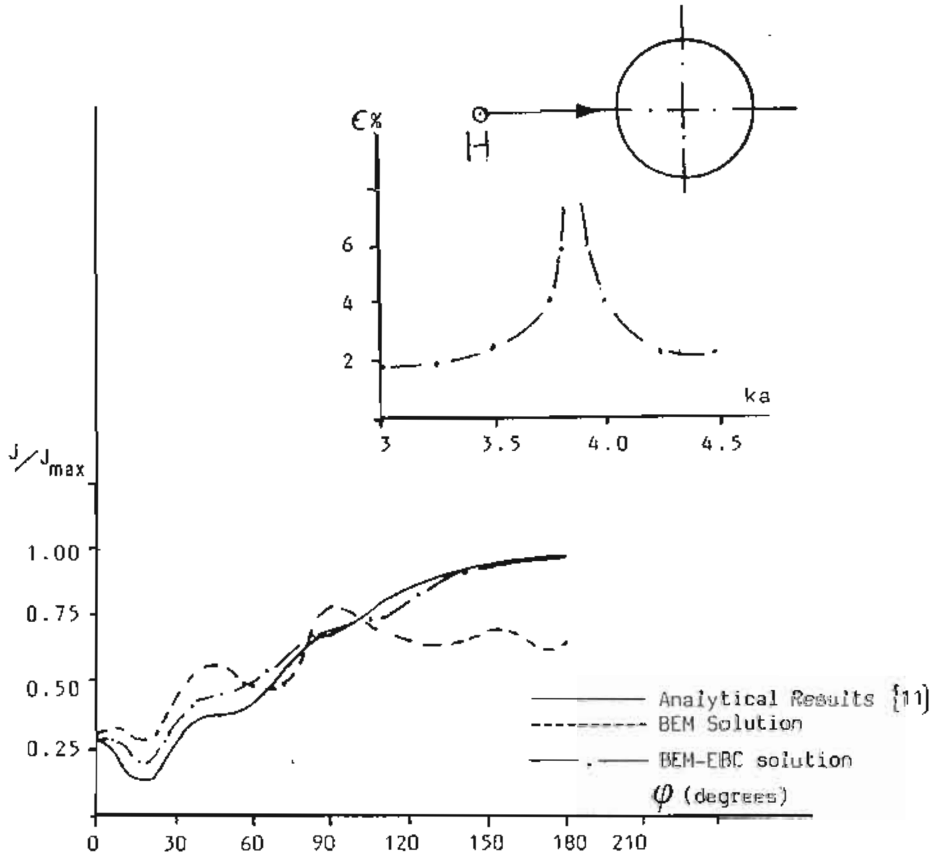


Fig. 1: Normalized current density over a circular cylinder at internal resonance.

Inset: Variation of accuracy check parameter ϵ near internal resonance. (BEM results).

which is often called the optical theorem. Thus, following [3] the quantity

$$\epsilon = \frac{4 \, \mathrm{Re}\left[F(\varphi^1)\right]}{\frac{1}{N}\sum_{n=1}^{N}\left[F(\varphi_n)\right]} - 1 \qquad .....(16)$$

has been used to check the energy balance. Away from resonance, 36 boundary elements have been found sufficient to achieve a value of ϵ less than 0.02 (Fig. 1). Howver, ϵ grows up rapidly as resonance is approached and using a finer grid does not produce any noticeable improvement. Similar findings have been reported in[3]. Fig. 1 shows the analytical and computed current distribution over the conducting cylinder. The variations in the conventional BEM solution are very different from those of the analytical solution. The use of the EBC with 8 interior constraint points results in a significant improvement.

V- Concluding Remarks

The present boundary element-conjugate gradient technique provides a powerful memory economic method which makes the solution of electromagnetic scattering problems feasible even with limited computer memory. The method is readily applicable to the analysis of scattering from a single conducting cylinder or from a number of parallel cylinders. However, the method trades memory space for execution time and with large problems computing times may be too long in spite of the fast convergence of the conjugate gradient technique. It may be possible by adopting an appropriate matrix preconditioning technique to achieve faster rate of convergence and consequently, to reduce the computation time. This possibility as well as the application of the method to scattering by dielectric cylinders are now under study.

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