

TORSION OF AN ELASTIC NON-HOMOGENEOUS HALF-SPACE HAVING A PENNY-SHAPED CRACK

by

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Abstract In this paper we study the effect of non-homogeneity on the stress concentration factor at the edge of a penny-shaped crack situated in a non-homogeneous elastic half-space parallel to its surface. The half-space is twisted by means of a circular rigid stump clamped to its surface. The stump's axis coincides with the crack's axis. The shear modulus is assumed to vary with two cylindrical coordinates r, z by a power law in the form $\mu = \mu_{\alpha,\beta} r^\alpha (c+z)^\beta$. Using the Hankel integral transform, the problem reduces to a pair of dual integral equations, the solution of which is governed by a pair of coupled Fredholm integral equations of the second kind for two auxiliary functions. The quantities of physical importance are expressed in terms of these functions. This pair of equations is solved numerically.

The coefficient of stress concentration at the rim of the crack is plotted as a function of the ratio of the stump's radius to that of the crack for different values of the crack's depth.

The results show that non-homogeneity appreciably reduces the coefficient of stress concentration.

1 Introduction

The torsional problem of an elastic half-space without crack by means of a rigid stump (the standard Reissner-Sagoci problem) was investigated by many authors. In [1]-[7] the half-space is assumed to be homogeneous but in [8]-[12] it is taken to be non-homogeneous, which is a case of geological importance since it may modelize a part of the Earth's crust. Similar problems for large thick plates are considered in [13]-[16].

The problem of homogeneous half-space with circular crack was studied in [17]-[18]. The problem of non-homogeneous half-space with penny-shaped flaws was studied in [19].

In the present paper, we consider the half-space to be non-homogeneous. The shear modulus is taken in the form $\mu_{\alpha,\beta} r^\alpha (c+z)^\beta$ where $\alpha \geq 0, \beta, c = 0, \mu_{\alpha,\beta}$ are constants and r is the radial coordinate of the cylindrical system of coordinates with origin at the center of the stump and z -axis oriented in the half-space. The crack's surface ($z = h, 0 \leq r \leq a$) is assumed to be stress free. The circular area ($0 \leq r \leq b$) of the boundary surface ($z = 0$) is forced to rotate through an angle Ω about the z -axis but the part of the boundary surface which lies outside this circle is stress free.

The problem reduces to that of determining two auxiliary functions from a pair of coupled Fredholm integral equations of the second kind. These are further reduced to a finite system of linear algebraic equations.

We have calculated some numerical values for the dimensionless concentration factor for different values of the geometrical parameters and of the inhomogeneity factor. The obtained results show that the concentration factor decreases as the degree of inhomogeneity increases. This is obviously a result of practical interest.

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2 Formulation of the problem

We consider the non-homogeneous elastic half-space ($r \geq 0, z \geq 0$) for which the circular area ($0 \leq r \leq b$) of its surface ($z = 0$) is forced to rotate about the z -axis. The part ($z = 0, b < r < \infty$) is assumed to be stress free Fig. (1). The half-space is slackened by a circular crack ($z = h, 0 \leq r \leq a$) assumed to be stress free. The shear modulus is taken in the form

$$\mu = \mu_{\alpha, \beta} r^{\alpha} (c + z)^{\beta}, \quad (1)$$

where $\alpha \geq 0$. *β. c.* $\mu_{\alpha, \beta}$ are constants.

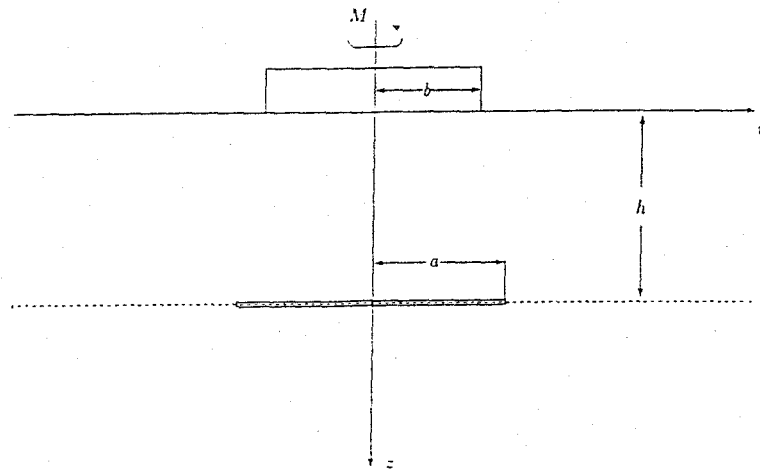


Figure 1: Formulation of the problem.

For the torsional problems of bodies of revolution, the only non-vanishing component of the displacement vector is the θ component $v(r, z)$, which is independent of θ . In the case of non-homogeneous and isotropic bodies this component must satisfy the equation [16]

$$\frac{\partial^2 v}{\partial r^2} + \frac{1 + \alpha}{r} \frac{\partial v}{\partial r} - \frac{1 + \alpha}{r^2} v + \frac{\beta}{c + z} \frac{\partial v}{\partial z} + \frac{\partial^2 v}{\partial z^2} = 0. \quad (2)$$

The only non-vanishing stress components $\tau_{\theta r}, \tau_{\theta z}$ are related to v through the equations :

$$\tau_{\theta r}(r, z) = \mu \frac{\partial}{\partial r} \left(\frac{v}{r} \right) \quad \text{and} \quad \tau_{\theta z}(r, z) = \mu \frac{\partial v}{\partial z}. \quad (3)$$

The following boundary conditions holds :

$$\tau_{\theta z}^{(1)}(r, 0) = 0, \quad (r > b) \quad (4)$$

$$v^{(1)}(r, 0) = \Omega r, \quad (0 \leq r \leq b) \quad (5)$$

$$\tau_{\theta z}^{(1)}(r, h) = \tau_{\theta z}^{(2)}(r, h) = 0, \quad (0 \leq r \leq a) \quad (6)$$

as well as the continuity equations

$$v^{(1)}(r, h) = v^{(2)}(r, h), \quad (r > a) \quad (7)$$

$$\tau_{\theta z}^{(1)}(r, h) = \tau_{\theta z}^{(2)}(r, h), \quad (r > a) \quad (8)$$

where quantities referring to the region ($0 \leq z \leq h$) are denoted by index ⁽¹⁾ and to the region ($z > h$) by ⁽²⁾.

3 Reduction to a pair of integral equations

Using the Hankel transform, the solution of equation (2) for the two regions takes the form

$$v^{(1)}(r, z) = r^{1-\nu}(c+z)^p \int_0^\infty \{A(\lambda)K_p[(c+z)\lambda] + B(\lambda)I_p[(c+z)\lambda]\} J_\nu(\lambda r) d\lambda, \quad (9)$$

$$v^{(2)}(r, z) = r^{1-\nu}(c+z)^p \int_0^\infty C(\lambda)K_p[(c+z)\lambda] J_\nu(\lambda r) d\lambda, \quad (10)$$

where $\nu = 1 + \frac{\alpha}{2}$, $p = \frac{1-\beta}{2}$, $J_\nu(x)$ is the Bessel function of the first kind of order ν . $I_p(x)$, $K_p(x)$ are the modified Bessel functions of the first and second kind, respectively, of order p and $A(\lambda)$, $B(\lambda)$ and $C(\lambda)$ are functions to be found.

Using conditions (6), (8) we get the following relations:

$$C(\lambda) K_{p-1}[\lambda(c+h)] = A(\lambda)K_{p-1}[\lambda(c+h)] - B(\lambda)I_{p-1}[\lambda(c+h)]. \quad (11)$$

The remaining conditions give the following pair of dual integral equations:

$$\int_0^\infty C_1(\lambda)J_\nu(\lambda r) d\lambda = f_1(r), \quad (0 \leq r \leq b) \quad (12)$$

$$\int_0^\infty \lambda C_1(\lambda)J_\nu(\lambda r) d\lambda = 0, \quad (r > b) \quad (13)$$

$$\int_0^\infty \lambda C_2(\lambda)J_\nu(\lambda r) d\lambda = f_2(r), \quad (0 \leq r \leq a) \quad (14)$$

$$\int_0^\infty C_2(\lambda)J_\nu(\lambda r) d\lambda = 0, \quad (r > a) \quad (15)$$

where

$$f_1(r) = \Omega r^\nu - \int_0^\infty C_1(\lambda) \left[\frac{K_p(\lambda c)}{K_{p-1}(\lambda c)} - 1 \right] J_\nu(\lambda r) d\lambda - \frac{\sqrt{c+h}}{c^{1-p}} \int_0^\infty C_2(\lambda) \frac{K_{p-1}[\lambda(c+h)]}{K_{p-1}(\lambda c)} J_\nu(\lambda r) d\lambda. \quad (16)$$

$$f_2(r) = \frac{2\sqrt{c+h}}{c^p} \int_0^\infty \lambda C_1(\lambda) \frac{K_{p-1}[\lambda(c+h)]}{K_{p-1}(\lambda c)} J_\nu(\lambda r) d\lambda - 2 \int_0^\infty \lambda^2(c+h)K_{p-1}[\lambda(c+h)] \left\{ \frac{I_{p-1}(c\lambda)K_{p-1}[\lambda(c+h)]}{K_{p-1}(\lambda c)} - I_{p-1}[\lambda(c+h)] + \frac{1}{2\lambda(c+h)K_{p-1}[\lambda(c+h)]} \right\} C_2(\lambda)J_\nu(\lambda r) d\lambda. \quad (17)$$

$$C_1(\lambda) = c^\nu [A(\lambda) K_{p-1}(\lambda c) - B(\lambda) I_{p-1}(\lambda c)], \quad (18)$$

$$C_2(\lambda) = \frac{B(\lambda)}{\lambda\sqrt{c+h} K_{p-1}[\lambda(c+h)]} \quad (19)$$

Using the fraction integration method [19], we get the solution of the last pair of dual integral equations in the form:

$$C_1(\lambda) = \sqrt{\lambda} \int_0^b \sqrt{t} \Phi(t) J_{\nu-1/2}(\lambda t) dt, \quad (20)$$

$$C_2(\lambda) = \sqrt{\lambda} \int_0^a \sqrt{t} \Psi(t) J_{\nu+1/2}(\lambda t) dt, \quad (21)$$

where the two auxiliary functions $\psi(t)$, $\phi(t)$ are determined from the following two integral equations, which can be obtained by substituting (20), (21) into (12), (14) :

$$\Phi(x) + \int_0^b \Phi(u) K_1(u, x) du + \int_0^a \Psi(u) L(u, x) du = \frac{\sqrt{2} \Gamma(\nu+1)}{\Gamma(\nu+1/2)} \Omega x^\nu, \quad (0 \leq x \leq b), \quad (22)$$

$$\Psi(x) - 2c^{1-2p} \int_0^b \Phi(u) L(x, u) du + \int_0^a \Psi(u) K_2(u, x) du = 0, \quad (0 \leq x \leq a), \quad (23)$$

and

$$L(u, x) = \frac{\sqrt{xu(c+h)}}{c^{1-p}} \int_0^\infty \lambda \frac{K_{p-1}[\lambda(c+h)]}{K_{p-1}(\lambda c)} J_{\nu-1/2}(\lambda x) J_{\nu+1/2}(\lambda u) d\lambda, \quad (24)$$

$$K_1(u, x) = \sqrt{ux} \int_0^\infty \lambda \left[\frac{K_p(\lambda c)}{K_{p-1}(\lambda c)} - 1 \right] J_{\nu-1/2}(\lambda x) J_{\nu-1/2}(\lambda u) d\lambda. \quad (25)$$

$$K_2(u, x) = -2\sqrt{ux} \int_0^\infty \lambda^2 (c+h) K_{p-1}[\lambda(c+h)] \left\{ \frac{I_{p-1}(c\lambda) K_{p-1}[\lambda(c+h)]}{K_{p-1}(\lambda c)} - I_{p-1}[\lambda(c+h)] + \frac{1}{2\lambda(c+h) K_{p-1}[\lambda(c+h)]} \right\} J_{\nu+1/2}(\lambda u) J_{\nu+1/2}(\lambda x) d\lambda. \quad (26)$$

4 Formulae for quantities of physical importance

Quantities of physical importance may be expressed in terms of functions $\Phi(x)$ and $\Psi(x)$.

The shear stress $\tau_{\theta z}$ under the stump is given by

$$\tau_{\theta z}(r, 0) = \mu_{\alpha, \beta} \sqrt{\frac{2}{\pi}} c^{1-2p} r^{2\nu-2} \frac{d}{dr} \int_r^b \frac{t^{1-\nu} \Phi(t) dt}{\sqrt{t^2 - r^2}}. \quad (r < b). \quad (27)$$

The torque M applied to the stump is calculated from the formula

$$M = -2\pi \int_0^b r^2 \tau_{\theta z}(r, 0) dr.$$

whence

$$M = 2\sqrt{\frac{2}{\pi}} \mu_{\alpha, \beta} c^{1-2p} \frac{\Gamma(\nu+1)}{\Gamma(\nu+1/2)} \int_0^b r^\nu \Phi(r) dr. \quad (28)$$

The shear stress $\tau_{\theta z}(r, h)$ at the edge of the crack takes the form

$$\tau_{\theta z}(r, h)|_{r=a} \approx \frac{K_a}{\sqrt{r^2 - a^2}} + O(1), \quad (29)$$

where K_a is the stress concentration factor at the rim of the crack given by

$$K_a = \lim_{r \rightarrow 0} \sqrt{r^2 - a^2} \tau_{\theta z}(r, h) = -\mu_{\alpha, \beta} \sqrt{\frac{1}{2\pi}} (c+h)^{\nu-1} a^{\nu-1} \Psi(a). \quad (30)$$

We shall be interested in the dimensionless factor

$$-\frac{a^2 K_a}{M} = \frac{c^{2\nu-1} (c+h)^{\nu-1} \Gamma(\nu+1/2)}{4 \pi^{3/2} \Gamma(\nu+1)} \frac{a^{\nu-1} \Psi(a)}{\int_0^b t^\nu \Phi(t) dt}. \quad (31)$$

Also,

$$\frac{b^2 K_b}{M} = \frac{-\Gamma(\nu+1/2)}{2 \pi^{3/2} \Gamma(\nu+1)} \frac{b^{\nu-1} \Phi(b)}{\int_0^b \tau^\nu \Phi(\tau) d\tau}, \quad (32)$$

where K_b is the concentrated stress intensity factor at the rim of the stump.

5 Special cases

In this section we derive results for some special cases.

CASE I: Letting $\alpha = \beta = 0$ and $a \neq 0$, one gets the corresponding problem for the homogeneous half-space having a penny-shaped crack. The two Fredholm integral equations (22) and (23) now take the form

$$\Phi(x) + \int_0^a \Psi(u) L^{(I)}(u, x) du = 2 \sqrt{\frac{2}{\pi}} \Omega x^\nu, \quad 0 \leq x \leq a, \quad (33)$$

$$\Psi(x) - 2 \int_0^b \Phi(u) L^{(I)}(x, u) du + \int_0^a \Psi(u) K_2^{(I)}(u, x) du = 0, \quad (0 \leq x \leq a), \quad (34)$$

and

$$L^{(I)}(u, x) = \sqrt{xu} \int_0^\infty \lambda e^{-\lambda h} J_{1/2}(\lambda x) J_{3/2}(\lambda u) d\lambda, \quad (35)$$

$$K_2^{(I)}(u, x) = \sqrt{xu} \int_0^\infty \lambda e^{-2\lambda h} J_{3/2}(\lambda x) J_{3/2}(\lambda u) d\lambda, \quad (36)$$

which is in agreement with the results of [18].

CASE II: Taking $\beta = 0$, one gets the corresponding problem for the nonhomogeneous half-space having a penny-shaped crack with nonhomogeneity depending on the radial coordinate by the power law $\mu = \mu_0 r^\alpha$.

Equations (22) and (23) take the form

$$\Phi(x) + \int_0^a \Psi(u) L^{(II)}(u, x) du = \frac{\sqrt{2}\gamma(\nu+1)}{\gamma(\nu+1/2)} \Omega x^\nu, \quad (0 \leq x \leq b), \quad (37)$$

$$\Psi(x) - 2 \int_0^b \Phi(u) L^{(II)}(x, u) du + \int_0^a \Psi(u) K_2^{(II)}(u, x) du = 0, \quad (0 \leq x \leq a). \quad (38)$$

and

$$L^{(II)}(u, x) = \sqrt{xu} \int_0^\infty \lambda e^{-\lambda x} J_{\nu-1/2}(\lambda x) J_{\nu+1/2}(\lambda u) d\lambda, \quad (39)$$

$$K_2^{(II)}(u, x) = \sqrt{xu} \int_0^\infty \lambda e^{-2\lambda x} J_{\nu+1/2}(\lambda x) J_{\nu+1/2}(\lambda u) d\lambda, \quad (40)$$

CASE III : Letting $a \rightarrow 0$, the crack is removed from the half-space.

The solution of the problem then reduces to that of a nonhomogeneous half-space twisted by a rigid disc. Equation (22) reduces to :

$$\Phi(x) + \int_0^b \Phi(u) K_1^{(III)}(u, x) du = \frac{\sqrt{2} \Gamma(\nu+1)}{\Gamma(\nu+1/2)} \Omega x^\nu, \quad (0 \leq x \leq b), \quad (41)$$

where

$$K_1^{(III)}(u, x) = \sqrt{ux} \int_0^\infty \lambda \left[\frac{K_\nu(\lambda c)}{K_{\nu-1}(\lambda c)} - 1 \right] J_{\nu-1/2}(\lambda x) J_{\nu-1/2}(\lambda u) d\lambda, \quad (42)$$

which is in agreement with the results of [10] for $\nu = 1$.

6 Numerical results and discussion

We have investigated special case II in detail numerically. The two Fredholm integral equations (37) and (38) were reduced to a finite system of linear algebraic equations. Numerical results for the dimensionless stress concentration intensity factor and for the dimensionless torque have been obtained for various values of the physical and the geometrical parameters as shown on figs. (2) - (6).

We have noticed that for the investigated interval $0 \leq a/b \leq 2.5$ the stress intensity factor generally decreases with increasing the inhomogeneity factor α . This is clear for

- (i) Small values of the dimensionless crack radius a/b .
- (ii) Large values of the dimensionless crack depth h/b .

However, for small crack depths and large crack radii, this feature could not be detected from the curves.

As for the dimensionless torque needed to produce the given angle of rotation, we have remarked that the influence of the crack becomes weaker as the inhomogeneity factor α increases a fact that is compatible with the physical sense.

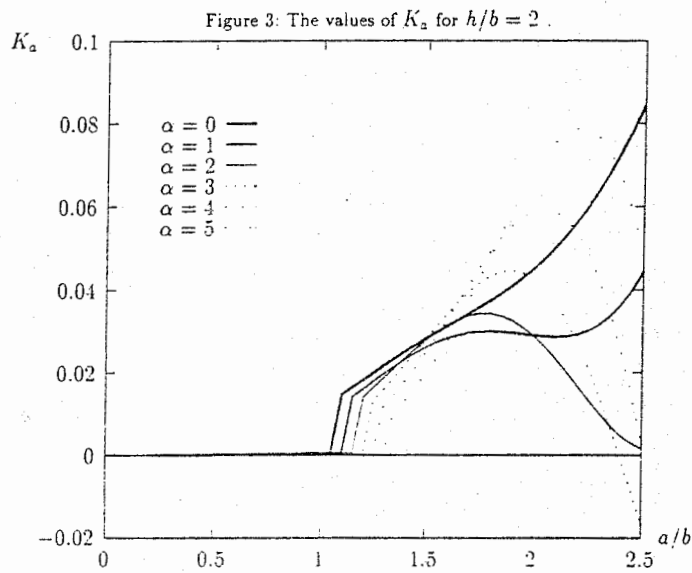
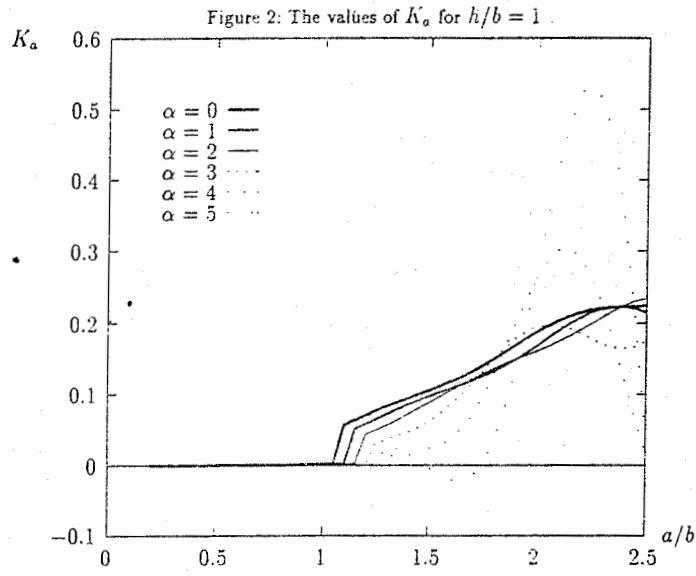
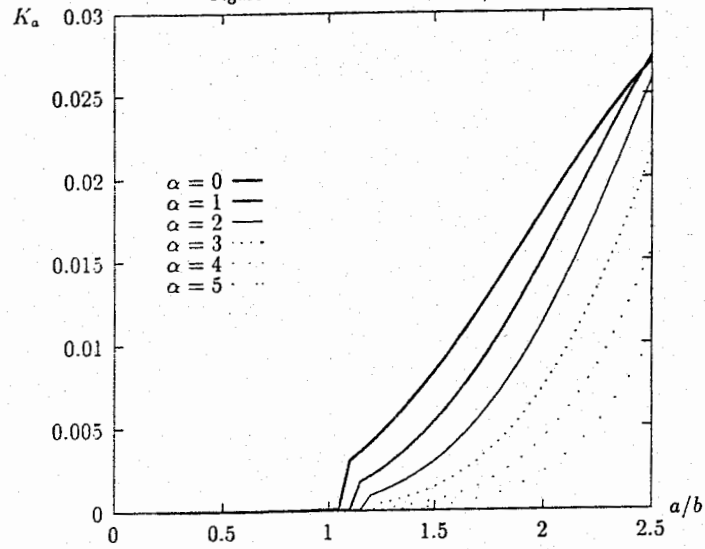
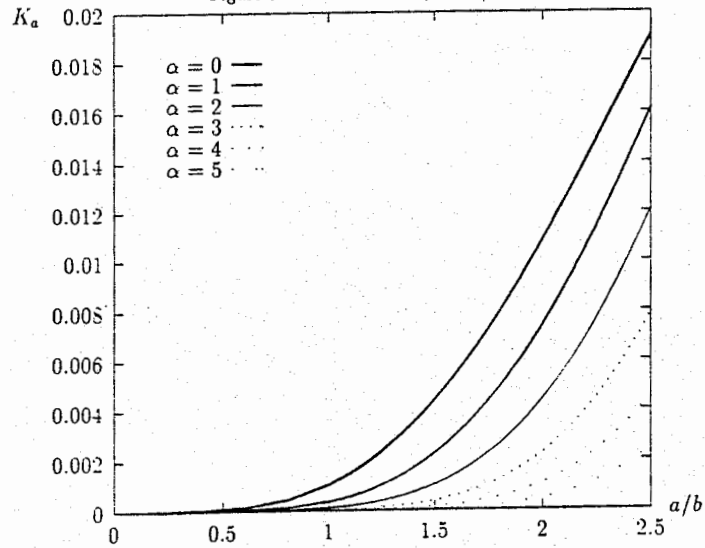
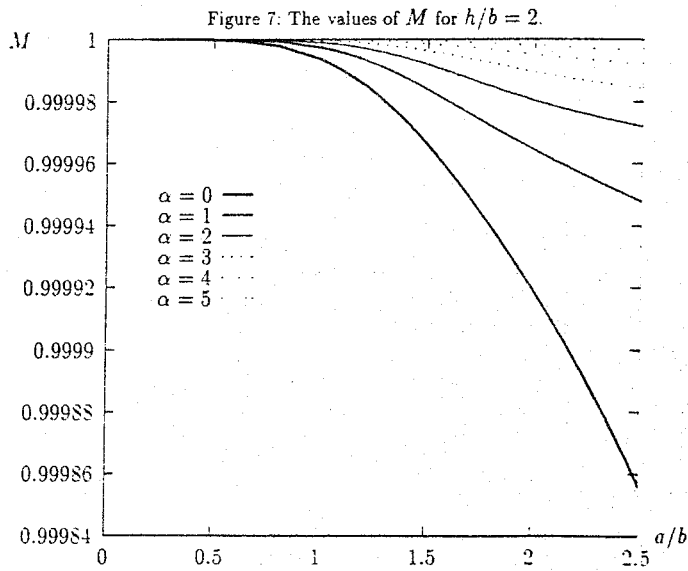
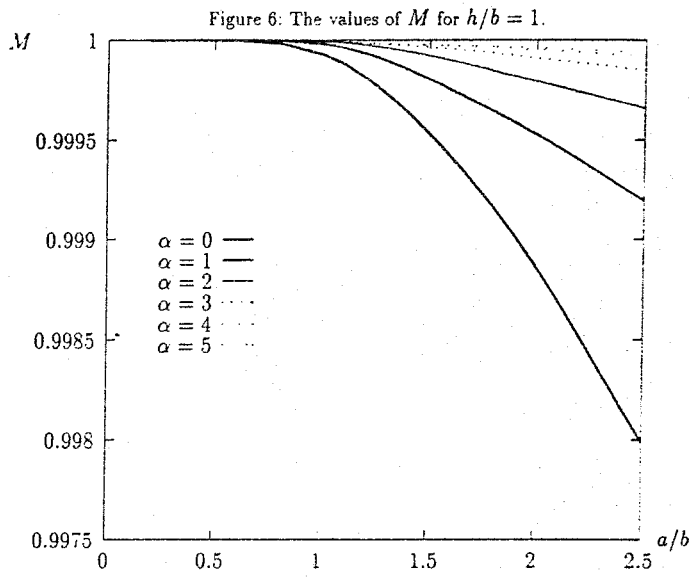
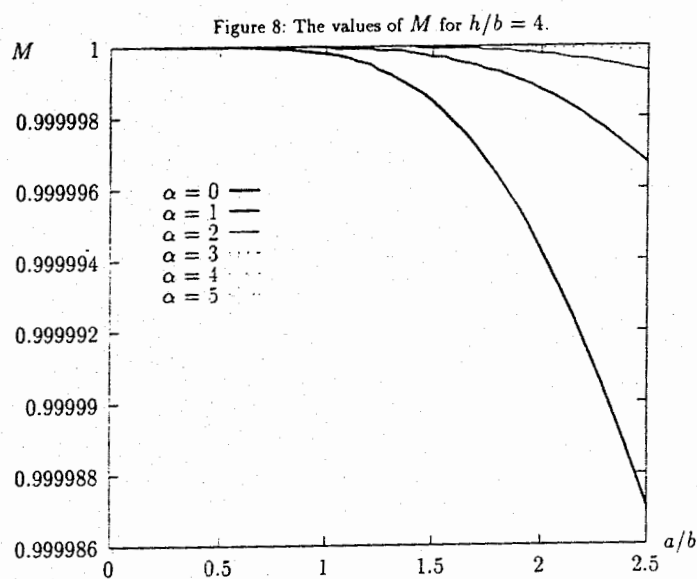


Figure 4: The values of K_a for $h/b = 4$.Figure 5: The values of K_a for $h/b = 5$.





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لى نصف فراغ مرن غير متجانس يحوى شرخ دائرى

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ملخص البحث

هذا البحث يدرس تأثير عدم التجانس على معامل تركيز الإجهادات عند حافة شرخ دائرى واقع داخل جسم مرن غير متجانس نصف لانهاى موازيا لسطحه . يقع نصف الفراغ تحت تأثير لى بواسطة قرص دائرى صلب مثبت على سطح النصف فراغ بحيث ينطبق كل من محورى الشرخ والقرص الصلب . معامل القص μ يتغير مع الإحداثيات الاسطوانية r, z تبعا لقانون القوى $\mu = \mu_0 r^a (c+z)^b$. باستخدام تحويلة هانكل التكاملية تؤول هذه المسألة إلى زوج من المعادلات التكاملية الآتية وهى بدورها تتحول إلى زوج من معادلات فريدهولم التكاملية من النوع الثانى فى دالتين مساعدتين . قد أمكن الحصول على صيغ لبعض الكميات ذات الأهمية الفيزيائية بدلالة هذه الدوال . وحلت معادلات فريدهولم عدديا . قد مثلت النتيجة فى صورة منحنيات تبين تغير معامل تكثيف الإجهاد عند حافة الشرخ مع النسبة بين نصفى قطر الشرخ والقرص وذلك لقيم مختلفة لعمق الشرخ . من الحسابات يمكن إستنتاج أن عدم التجانس يقلل بصورة ملحوظة من تكثيف الإجهادات .