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ON THE OSCILLATORY BEHAVIOR OF SOLUTIONS OF NON LINEAR SECOND ORDER DIFFERENTIAL EQUATION

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ABSTRACT

Oscillation criteria are given for the second order nonlinear differential equation

$$(a(x)h(y(x)g(y'(x)))' + P(x)f(y(x)) = 0$$

and the generalized Euler-type functional equation

$$(a(x)h(y(x)g(y'(x))) + P(x)f(y(q(x))) = 0$$

1. INTRODUCTION

Since the well known papers of Kamenev [4] , [5] 1970's were published, a great number of papers were devoted to study the oscillatory behavior of second order differential equations using integral criterion (see for example [9], [10], [12], [13]. In [1], Chen and Yeh were able to extend Kamenev's results for the second order differential equation.

$$(a(x)y'(x))' + P(x)f(y(x)) = 0 \qquad \text{.............. (1.1)}$$

Meanwhile, Mahfoud [7] discussed the case of differential functional equation

$$(a(x)y'(x))' + P(x)f(y(q(x))) = 0 \qquad \text{.............. (1.2)}$$

which generalize the results on Euler-type delay equation by Opial [8] and Wong [11]. In this paper we are concerned with oscillatory behavior of solutions of the more general differential equations

$$(a(x)\,h(y(x))\,g(y'(x))' + P(x)\,f(y(x) = 0 \quad \text{..............} \quad (1.1)$$

and

$$(a(x(h(y(x))\,g(y'(x)))' + P(x)\,f(y(q(x))) = 0 \quad \text{..............} \quad (1.2)$$

In section 2, we discuss the oscillatory behaviour of (1.1) using Kamenev's integral criterial [1], [4]. In section 3, we study oscillation results for the functional differential eqution (1.2). The obtained results extend those of [7], [8] and [11].

In what follows, we consider oly such solutions which are defined for all x $\geq x_0 \geq 0$. The oscillatory character is considered in the usual sense, i.e., a continuous real-valued function y defined on [xy, ∞], for some $x_y \geq 0$, is called oscillatory if its set of zeros is unbounded above, otherwise it is called non oscillatory.

2. Kamenev's integral criteria

Consider the differential equation

$$(a(x)\,h(y(x))\,g(y'(x)))' + P(x)\,f(y(x)) = 0 \quad \text{..............} \quad (1.1)$$

where a,p : $[x_0, \infty] \to R$ and h, g, f : $R \to R$. We assume that the functions appearing in (1.1) be sufficiently smooth for a local existence and uniqueness theoren to hold for (1.1) for $x \in (x_0, \infty]$. We suppose the following hypothesses :

(H_1) $a(x) > 0$, $p(x) > 0$, $f(y(x)) > 0$ and $h(y(x)) \geq c > 0$ for all $x \geq x_0 \geq 0$

(H_2) $\int^{\infty} p(x)\,dx$ esists.

$$(H_3)\ \int^{\infty} \frac{dy(s)}{a(s)\,h(y(s)\,g(y'(s))} = \infty$$

The following lemma is needed for our results.

Lemma 2.1 Let the hypothesses (H_1), (H_2), (H_3) and $H_4)$ be satisfied. Define

$$W(x) = \frac{a(x)\,h(y(x))\,g(y'(x))}{f(y(x))} \qquad \text{.............. (2.1)}$$

If $Y(x) \neq 0$ be a nonoscillatory solution of (1.1), then

$$V(x) = \int_x^{\infty} \frac{W^2(s)\,f'(y(s))\,dy(s)}{a(x)\,h(y(s))\,g(y'(s))} < \infty \qquad \text{.............. (2.2)}$$

Proof. By (1.1) and (2.1), it follows that

$$W'(x) + \frac{W^2(x)\,f'(y(x))\,y'(x)}{a(x)\,(h(s))\,g(y'(x))} + P(x) = 0 \qquad \text{.............. (2.3)}$$

Integrating (2.3) from x to τ, we obtain

$$W(\tau) - W(x) + \int_x^{\tau} \frac{W^2(s)\,f'(y(s))\,dy(s)}{a(s)\,h(y(s))\,g(y'(s))} = -\int_x^{\tau} P(s)\,ds \qquad \text{....... (2.4)}$$

Thus by (H_2),

$$\lim_{\tau\to\infty} \left[W(\tau) + \int_x^{\tau} \frac{W^2(s)\,f'(y(s))\,dy(s)}{a(s)\,h(y(s))\,g(y'(s))} \right] = K \text{ exists and is finite.} \qquad (2.5)$$

Now, if (2.2) does not hold, then by (2.5),

$$\lim_{\tau \to \infty} \frac{W(\tau)}{\int_x^\tau \frac{W^2(s) f'(y(s))\, dy(s)}{a(s)\, h(y(s))\, g(y'(s))}} = -1$$

This means that there exists $x_1 \geq x_o$ such that for $\tau \geq x_1$, there exists a positive constant $k_1 < 1$ such that,

$$\frac{W(\tau)}{\int_x^\tau \frac{W^2(s) F'(y(s))\, d\,y(s)}{a(s)\, h(s))\, g(y'(s))}} \leq -k_1 \qquad \text{.............. (2.6)}$$

Thus by (H_4), we have for $x_1 \geq x_o$,

$$\frac{Ky'(\tau)}{K_1^2\, a(\tau)\, h(y(\tau))\, g(y'(\tau))} \leq \frac{f'(y(\tau)\, y'(\tau)}{K_1^2\, a(\tau)\, h(y(\tau))\, g(y'(\tau))}$$

$$\leq \frac{W^2(\tau)\, f'(y(\tau))\, y'(\tau)}{a(\tau)\, h(y(\tau))\, g(y'(\tau))} \left[\int_{x1}^z \frac{W(s)\, f(y(s))\, dy(s)}{a(s)\, h(y(s))\, g(y'(s))}\right]^{-2} \qquad \text{........ (2.7)}$$

i.e.,

$$\int_{x_1}^z \frac{k\, dy(\tau)}{x_1\, K_1^2\, a(\tau)\, h(y(\tau))\, g(y'(\tau))} \leq \int_{x_1}^z \frac{W^2(\tau)\, f'(y(\tau))\, y'(\tau)}{a(\tau)\, h(y(\tau)\, g(y'(\tau))}$$

$$\left[\int_{x_1}^z \frac{W'(s)\, f'(y(s))\, dy(s)}{a(s)\, h(y(s))\, g(y'(s))}\right]^{-2} d\tau \leq \int_{x_1}^z \frac{du}{u^2} \qquad \text{........ (2.8)}$$

Thus,

$$\frac{K}{K_1^2} \int_{x_1}^\infty \frac{dy(s)}{a(s)\, h(y(s))\, g(y'(s))} \leq \int_{x_1}^\infty \frac{du}{u^2} < \infty$$

This contradicts (H_3),

Remark : The above theorem includes the result of [1] for the case $g(y') = y'$ and $h(y) = y$. Moreover, if $a(x) = 1$ the theorem includes that of Hartmant [3].

Theorem 2.2 : Let the assumption $(H_1) \rightarrow (H_3)$ and (H_4) be satisfied. Let

$$\rho(x) = \int_x^{\infty} P(s)\, ds,\ A(x) = \exp\left(-4k \int_{x_\circ}^{x} \frac{\rho(s)\, ds}{a(s)\, h(y(s))}\right)$$

$$B(x) = \int_{x_\circ}^{x} A(s)\, ds,\ \phi(x) = \int_{x_\circ}^{x} \frac{dy(t)}{f(t)} > 0. \text{ If } g(y') \leq y',$$

Then one of the two conditions

$$(C_1) \lim_{x\to\infty} \sup \left\{ \left[B(x) \left(B(x) + \int_{x_\circ}^{x} \frac{a(s)}{s} ds\right) \right]^{\frac{1}{2}} / \int_{x_\circ}^{x} \rho(s)\, ds / \right\} = \infty$$

or

$$(C_2) \lim_{x\to\infty} \inf \int_{x_\circ}^{x} \phi(s) \left[\rho(s) + CA(s) = -\infty,\ C > 0 \right.$$

with $a'(x) \leq 0$ and $\phi(x) \leq 0$

is sufficient for (1.1) to be oscillatory.

Proof. Assume that $y(x) \neq 0$, $x \geq x_0$ be a nonoscillatory solution of (1.1). Since by (2.1) and (2.2).

$$W(x) = \frac{a(x)\, h(y(x))\, g(y'(x))}{f(y(x))},$$

and

$$V(x) = \int_x^{\infty} \frac{W^2(s)\, f'(y(s))\, dy(s)}{a(s)\, h(y(s))\, g(y'(s))}.$$

Then by lemma 2.2

$$V(x) < \infty \qquad \text{............ (2.9)}$$

Now by (2.5) and (2.9), it follows that

$$\lim_{x \to \infty} W(x) = 0 \qquad \text{............. (2.10)}$$

Substituting from (2.10) into (2.4) after taking the limits, it follows that,

$$W(x) = V(x) + \rho(x) \qquad \text{............. (2.11)}$$

Thus by (H_4) and (2.11) , we obtain

$$-V'(x) = \frac{W^2(x)\, f'(y(x))\, y'}{a(x)\, h(y(x))\, g(y')} = \frac{f'(y(x))\, [V(x) + \rho(x)]^2\, y'}{a(x)\, h(y(x))\, g(y'(x))} \geq \frac{4K\, V(x)\, \rho(x)}{a(x)\, h(y(x))}.$$

Hence,

$$V(x) \leq C \exp\{-4K \int_{x_o}^{x} \frac{\rho(s)\, ds}{a(s)\, h(y(s))}\} = C\, A(x). \qquad \text{.......... (2.13)}$$

Now by (H_4),

$$K \int_{x}^{\infty} \frac{W^2(s)\, dy(s)}{a(s)\, h(y(s))\, g(y'(x))} \leq \int_{x}^{\infty} \frac{W^2(s)\, f'(y(s))\, dy(s)}{a(s)\, h(y(s))\, g(y'(s))} \leq C\, A(x)$$

i.e,

$$\int_{x_o}^{x} d\tau \int_{x}^{\infty} \frac{W^2(s)\, dy(s)}{a(s)\, h(y(s))\, g(y'(s))} \leq K_1\, B(x). \qquad \text{.......... (2.14)}$$

By intetgrating by parts, it follows that,

$$(x - x_o) \int_{x}^{\infty} \frac{W^2(s)\, dy(s)}{a(s)\, h(y(s))\, g(y'(s))} + \int_{x_o}^{x} (s - x_o) \frac{W^2(s)\, dy(s)}{a(s)\, h(y(s))\, g(y'(s))} \leq K_1\, B(x) \qquad \text{............. (2.15)}$$

Thus,

$$\int_{x_o}^{x} \frac{s\, W^2(s)\, dy(s)}{a(s)\, h(y(s))\, g(y'(s))} - K_2 \leq K_1\, B(x) \quad \text{where } K_1 > 0 \text{ and } + K_2 > 0.$$

Now using Schwarz's inequality

$$\left(\int_{x_0}^{x} W(s)\, dy(s)\right)^2 \leq \left(\int_{x_0}^{x} \frac{sW^2(s)\, dy(s)}{a(s)\, h(y(s))\, g(y'(s))}\right) \left(\int_{x_0}^{x} \frac{a(s)\, h(y(s))\, g(y'(s))}{s} dy(s)\right)$$

$$\leq [K_2 + K_1 B(x)] \left(\int_{x_0}^{x} \frac{a(s)\, h(y(s))\, g(y'(s))}{s} dy(s)\right) \quad \text{......... (2.16)}$$

Thus by (2.11), (2.12) and (2.14), it follows that

$$\left|\int_{x_0}^{x} V(s)\, dy(s)\right| \leq \left\{ B(x) + \int_{x_0}^{x} \frac{a(s)\, h(y(s))\, g(y'(s))\, dy(s)}{s}\right\}^{\frac{1}{2}}$$

Thus,

$$\left|\int_{x_0}^{x} \rho(s)\, ds\right| \leq \left\{ B(x) + [K_2 + K_1 B(x)] \cdot \int_{x_0}^{x} \frac{a(s)\, h(y(s))\, g(y'(s))\, dy(s)}{s}\right\}^{\frac{1}{2}}$$

$$\leq [K_2 + K_1 B(x)] \left(\int_{x_0}^{x} \frac{a(s)\, h(y(s))\, g(y'(s))}{s} dy(s)\right)$$

i.e.,

$$\left\{B(x)\left[B(x) + \int_{x_0}^{x} \frac{a(s)\, h(y(s)\, g(y'm(s))\, dy(s)}{s}\right]\right\}^{-\frac{1}{2}} \cdot \left|\int_{x_0}^{x} \rho(s)\, ds\right| \leq K_3 \quad \text{.......... (2.17)}$$

This is a contradiction with (C_1)

Now let the condition (C_2) holds. By (2.11) and (2.13), it follows that,

$$\int_{x_0}^{x} \phi(s)\, W(s)\, dy(s) = \int_{x_0}^{x} \frac{\phi(s)\, a(s)\, h(y(s))\, g(y'(s))\, dy(s)}{f(y(s))}$$

$$= \int_{x_0}^{x} \phi(s)\, [V(s) + \rho(s)]\, dy(s) \leq \int_{x_0}^{x} \phi(s) + \rho(s)]\, dy(s)$$

Since, $\phi(s)\, a(s)\, h(y(s))\, g(y'(s)$ is positive, thus by the first mean value theorem,

$$\int_{x_o}^{x} \frac{\phi (s)\, a (s)\, h (y (s))\, g (y' (s))\, dy (s)}{f (y (s))} =$$

$$= \phi (\xi)\, a (\xi)\, h (y (\xi))\, g (y' (\xi)) \int_{x_o}^{x} \frac{dy (s)}{f (y (s))} \qquad \text{for } x_o \le \xi \le x$$

$$= \phi (\xi)\, a (\xi)\, h (y (\xi))\, g (y' (\xi))\; [\, \phi (y (x)) - \phi (y (x_o))].$$

But since $a' (x) \le 0$, $\phi (s)\, a (s)\, h (y (s))\, g (y' (s))$ is nonincreasing. So,

$$\int_{x_o}^{x} \frac{\phi (s)\, a (s)\, h (y (s))\, g (y' (s))\, dy (s)}{f (y (s))} \ge - \phi (x_o)\, a (x_o)\, h (y (x_o))\, g (y' (x_o))\, \phi (y (x_o)) \qquad \text{.......... (2.19)}$$

Thus by (2.1), (2.11) and (2.13), it follows that

$$- \phi (x_o)\, a (x_o)\, h (y (x_o))\, g (y' (x_o))\, \phi (y (x_o)) \;\le \int_{x_o}^{x} \phi (sd)\, [CA (s) + \rho (s)]\, dy (s) \rightarrow -\infty \text{ as } x \rightarrow +\infty$$

This contradicts (2.18).

3. A generalized Euler's type equation

Consider the differential equation

$(a (x)\, h (y (x))\, g (y' (x))) + p (x)\, f (y (q (s))) = 0$

where $a,p,q : [0, \infty) \rightarrow (-\infty, \infty)$ and $h,g,f : (-\infty,\infty) \rightarrow (-\infty,\infty)$.

We also assume that p,q,h and f are continuous and a is continuously differentiable with

$(a (x) > 0,\ q (x) \ge 0$ and $q (s) \le x$ for all $x \ge 0$.

Assume that

Since by assumption, the R.H.S. of the inequality (3.4) tends to $-\infty$, this is a contradiction with the assumption $y(x) > 0$. Thus $y'(x) \geq 0$ for $x \geq x_1$. This means that $y(x) \to \ell$ as $x \to \infty$, $0 < \ell < \infty$. But since by assumption f is continuous, we have :

$$f(y(q(x))) \to f(\ell) \quad , \text{as } x \to \infty.$$

Thus there exists $x^* \geq x_1$, such that

$$f(y(q(x))) \geq \frac{f(\ell)}{2} \qquad \text{for } x \geq x^*.$$

Now, multiplying both sides of (1.2) by $\mu(x)$ and integrating from x_2 to x and using the inequality (3.,6), we directily obtain

Again the assumption $(G(y') \geq Ky'$ reduces (3.7) to the form

$$\int_{x_2}^{x} \mu(s)\ (a(s)\, h(y(s))\, g(y'(s)))'ds + \frac{f(\ell)}{2} \int_{x_2}^{x} \mu(s)\, p(s)\, ds \leq c_1$$

using the integration by parts and discarding the nonnegative terms we obtain

$$\int_{x_2}^{x} \mu(s)\, p(s)\, ds \leq \varepsilon \qquad \text{.......... (3.8)}$$

for some constant ε : 0 and for all $x \geq x_2$. This contradicts (3.1).

The case $y(x) < 0$ and for all $x \geq x_2$. This contradicts (3.1).

Remark : The special case $h(y(x)) = 1$, $g(y') = y'$. Theorem 3.1 includes theorem 2 of |7|. Moreover, the theorem includes theorem B of Opial |8| if $q(x) = x$.

$\lim_{y\to\infty}$ (h (y) = ∞, yf (y) > 0 for y ≠ 0 and g (y') ≥ ky', k > 0.

Theorem 2.1 : Suppose that g (y') ≥ ky' and

$$\mu(x) = \int_o^x \frac{ds}{a(s)\, h(y(s))} \to \infty \quad \text{as } x \to \infty. \text{ If}$$

$$\int_o^x \mu(x)\, \rho(x)\, dx = \infty \qquad \text{.......... (3.1)}$$

then every bounded solution of (1.2) is oscillatory.

Proof. Assume the contrary that y (x) be a bounded nonoscillatory solution of (01.2). Suppose y (x) ≥ 0 for x ≥ xo, xo ≥ 0. now it is clear by (1.2), that

$$a(x)\, h(y(x)\, g(y'(x)))' \le 0. \qquad \text{.......... (3.2)}$$

i.e. a (x) h (y (x)) g (y' (x)) is nonincreasing for $x \ge x_1$. Hence since

g (y' (x)) ≥ k y' (x),

It follows that a (x) h (y (x)) y' (x) is also non increasing for $x \ge x_1$.

We first clain that y' (x) ≥ 0 for $x \ge x_1$. Suppose that it is false. Then there exists $\bar{x} \ge x_1$ such that y' $(\bar{x})$ < 0. Consequently

$$a(\bar{x})\, h(y(\bar{x}))\, y'(\bar{x}) = -c \quad , \quad c > 0 \qquad \text{.......... (3.3)}$$

Thus

$$y(x) \le y(\bar{x}) - c \int_{\bar{x}}^x \frac{ds}{a(s)\, h(y(s))} \text{ for } x \ge \bar{x} \qquad \text{.......... (3.4)}$$

Theorem 3.2 : Let $x \geq 0$. Assume that $g(y') \geq Ky'$, $K \geq 1$ and $\int_x^\infty \frac{ds}{a(s)\,h(y(s))} = \infty$. Suppose that there exists continuously differentiable functions.

$\xi, \eta : R^+ \; \{0\} \rightarrow R^+$ and $\rho : R \rightarrow R$

such that,

(H1) $q(x) \geq \rho(x)$ and $\rho'(x) > 0$ for $x \geq X$ and $\lim_{x\to\infty} \rho(x) = \infty$

(H2) $|f(y)| \geq |\xi(y)|$, $\xi'(y) \geq \varepsilon$, $\varepsilon > 0$ and $\lim_{x\to\infty} \rho(x) = \infty$

(H3) $\limsup \int_x^x [\, p(s)\,\eta(s) - \frac{\eta'^2(s)\,h(y(s))\,a(\rho(s))}{4\,\varepsilon\eta(s)\,\rho'(s)}\,]\, ds = \infty$

Then the equation (1.2) is oscillatory.

Proof. Suppose that there exists a nonoscillatory solution $Y(x) > 0$ of (1.2) for $x \geq xo$, $xo \geq 0$. Then as in the proof of theoren 3.1, it follows that there exists $x1 \geq xo$ such that $y(x)$ is non-decreasing for $x \geq x1$. Now choosing $x2 \geq x1$ such that $p(x)\, x1$ for $x \geq x2$, then by (1.2) and the assumptions, it follows that

$$(a(x)\,h(y(x))\,g(y'(x)))' + p(x)\,\xi(y(x)) \leq 0 \text{ for } x \geq x_2$$

Thus by assumption if $k \geq 1$, then

$$(a(x)\,h(y(x))\,g(y'(x)))' + p(x)\,\xi(y(x)) \leq 0 \text{ for } x \geq x_2 \quad \text{.......... (3.9)}$$

Multiplying both sides of (3.9) by $\frac{\xi(x)}{\xi(y(\rho(x)))}$ and integrating from x2 to x, we obtain

$$\int_{x_2}^{x} p(x)\,\eta(s)\,ds + \int_{x_2}^{x} \frac{a(s)\,h(y(s))\,y'^2(s)\,\xi'(y(\rho(s)}{\xi^2(y(\rho(s)))}\,ds - \int_{x_2}^{x} \frac{a(s)\,h(y(s))\,y'(s)\,\eta(s)}{\xi(y(\rho(s)))}\,ds \le c_1$$

But since by above, the quantity (a (x) h (y (s) y' (x)) is nonincreasing, it follows by the assumption $x \ge q(x)$ for all $x \ge 0$ and (H_1) that

$$a(x)\,h(y(x))\,y'(x) \le a(\rho(x)))\,y'(\rho(x)), \quad x \ge x_2$$

Thus by (H_2), we obtain

$$\int_{x_2}^{x} p(s)\,\eta(s)\,ds + \varepsilon \int_{x_2}^{x} \frac{a^2(s)\,h(y(s))\,y'^2(s)\,\eta(s)\,\rho'(s)}{a(\rho(s))\,\xi^2(y(\rho(s)))}\,ds - \int_{x_2}^{x} \frac{a(s)\,h(y(s))\,y'(s)\,\eta'(s)}{\xi(y(\rho(s)))}\,ds \le c$$

i.e.,

$$\int_{x_2}^{x} p(s)\,\eta(s)\,ds - \int_{x_2}^{x} \frac{a(\rho(s))\,h(y(s)\eta'^2(s)}{4\,\varepsilon\eta(s)\,\rho'(s)}\,ds + \varepsilon \int_{x_2}^{x} \frac{\eta(s)\,\rho'(s)\,h(y(s))}{a(\rho(s))}\left[\frac{a(s)\,y'(s)}{\xi(y(\rho(s)))} - \frac{a(\rho(s))\,\eta'(s)}{2\,\varepsilon\rho'(s)\,\eta(s)}\right]^2 ds \le c$$

Since the last integral in (3.10) is nonnegative,

$$\int_{x_2}^{x} \left[p(s)\,\eta(s) - \frac{a(\rho(s)\,h(ys))\,\eta'^2(s)}{4\,\varepsilon\eta(s)\,\rho'(s)}\right] ds \le c$$

This is a contradiction with the assumption (H_3). The case $y(x) < 0$ for $x \ge x_o$ is similar and the proof is completed.

Remark : 1- If ($\eta(x) = x$, $a(x) = 1$, $z(y) = y$, $e = 1$ and $r(x) = cx$. Then the above theorem includes theorem (A) of Wong |11|.

2- In a recent paper of Lalli |6|, an oscillation criteria was given for the differential equation (1.2) but the condition considered there,

$$\int \frac{ds}{a(s)} = \infty$$

is slightly stronger than ours.

3- The functional equation considered by Grace and Lalli |2| is much different from ours.

REFERENCES

[1] Lu-San Chen and Cheh-Chih Yeh, "Oscillation theorem for second order nonlinear differential equations with an integrally small coefficient". J. Math. Anal. and Appl. 78 (1980) 49 - 57.

[2] S. R. Grace and B. S. Lalli, "Oscillatory behavior of solutions of second order differential equations with alternating coefficients" Math. Nachr. 127 (1986) 165 - 175.

[3] P. Harmtman "Ordinary differential equations" Wiley-Interscience, New York, 1964.

[4] I. V. Kamenev, "Oscillation of solutions of a second order differential'nye Warnenye equation with an integrally small coefficient" Differential'nye 13 (1977), 2141 - 2148.

[5] I. V. Kamenev, "An integral criterion for oscillation of linear differential equations of second order", Math. Zametki 23 91978), 249 - 251 (in Russian); English translation; Math. Notes 23 (1978), 136 - 138.

[6] B. S. Lalli, "Integral averages and second order sublinear oscillations" Math. Nachr. 145 (1990) 223 - 231.

[7] W. E. Mahfoud, "Oscillation theorems for a second order delay differential equations" J. Math. Anal. Appl. 63 (1978), 339 - 346.

[8] Z. Opial, "Sure une critere d'oscillation de l'equation differentielle (Q (t) x') '+f(t) x=0, Ann. Polon. Math. 6 (1959), 99 - 104.

[9] Ch. G. Philos, "On a Kamenev's integral criterion for oscillation of linear differential equations of second order" Utilitas Mathematica, Vol. 224 91983), 227 - 289.

[10] Ch. G. Philos, "Oscillation of second order linear ordinary differential equations with alternating coefficients" Bull. Austral. Math. Soc. Vol. 27 (1983), 307 - 313.

[11] J. S. W. Wong, "Second order oscillation with retarded arguments, in ordinary differential equations". pp. 581 - 596, Academic press, New York - Londion 91972).

[12] Cheh-Chin Yeh, "An oscillation criterion for second order nonlinearl differential equations with functional arguments", J. Math. Anal. Appl. 76 (1980), 72 - 76.

[13] Cheh-Chin Yeh, "Oscillation theorems for nonlinear second order differential equations with damped term". Proceedings of the Americal Mathematical Society 84 91982) No. 3, 397 - 402.