Mansoura University Faculty of Engineering	First year Final Exam 2014	
Math. & Engineering Physics Dept.	Math. (4)	Time : 3 hours

[1]-(a) Find the eigenvalues and eigenfunctions of the Sturm-Liouville problem
(0.1)

$$y'' + \lambda y = 0, \qquad y(0) = y'(1) = 0. \qquad [10\text{ pts}]$$

[b] Find the Fourier series of the periodic function of period 2π

$$f(x) = \frac{x^2}{4}, \qquad |x| < \pi$$

and use this series to verify the identities

$$\frac{\pi^2}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots \qquad [10\text{ pts}]$$

[c] Prove that

$$4\,J_n''(x) - J_{n+2}(x) + 2\,J_n(x) - J_{n-2}(x) = 0 \qquad [5\text{ pts}]$$

[d] Evaluate

$$\int_{-\infty}^{\infty} \frac{e^{2x}}{a\,e^{3x} + b}\,dx \qquad \int_2^5 (x-2)^{1/2}(5-x)^{1/3}\,du \qquad [10\text{ pts}]$$

[2]-(a) Using the Fourier integral representation, show that

$$\int_0^{\infty} \frac{\cos wx + w \sin wx}{1+w^2}\,dw = \begin{cases} 0, & x < 0, \\ \pi/2, & x = 0 \\ \pi\, e^{-x}, & x > 0 \end{cases} \qquad [10\text{ pts}]$$

[b] Solve IBVP

$$u_t = u_{xx} - e^{-x}, \qquad 0 < x < 1, \quad t > 0$$

$$u(0,t) = u(1,t) = 10, \qquad u(x,0) = \eta(x) \qquad [10\text{ pts}]$$

[c] Show that

$$\frac{d}{dx}\left[x^n\, J_n(ax)\right] = a\, x^n\, J_{n-1}(ax) \qquad [5\text{ pts}]$$

[d] Obtain the Legendre Polynomial $P_4(x)$ from Legendre differential equation

$$(1-x^2)\,y'' - 2x\,y + n(n+1)\,y = 0. \qquad [10\text{ pts}]$$

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3. **(a)** Show that $u(x,y) = x + e^y \cos x$ is harmonic. Find an analytic function $f(z) = u(x,y) + iv(x,y)$ as a function of z. [10

(b) Show that $|\sinh y| \leq |\sin z| \leq \cosh y$. Describe the points z at which: [10 pts]

i. $|\sin z| = |\sinh y|$.

ii. $|\sin z| = \cosh y$.

(c) i. Let $z = x + iy$. Define each of the following: [10 pts]

$-z$, $\bar{z}$, $|z|$, $\operatorname{Arg} z$ and $\arg z$.

ii. Consider the n^{th} roots w_k of the complex number $z = -1$.

1. Write the algebraic properties of w_k.
2. Write the geometric properties of w_k.
3. There are some properties of w_k which depend on the evenness or oddness of n, write some of these properties.

4. **(a)** Evaluate the following without using the residue theorem: [10 pts]

i. $\int_C (e^{\cos z} \sin z + \bar{z})dz$, where c is the line segment from $z = 0$ to $z = \frac{\pi}{2} + i\frac{\pi}{2}$. Put your answer in the form $a + ib$.

ii. $\int_{|z|=2} \left(e^{z^2} + \frac{\cos z}{z(z-1)^3}\right) dz$.

(b) Use the residue theorem to evaluate [10 pts]

$$\int_{|z|=2} \left(\frac{\sin z}{z\left(z-\frac{\pi}{2}\right)^2} + \frac{3}{2}(z-1)^3 e^{\frac{1}{z-1}} \right) dz.$$

(c) Consider the function f given by $f(z) = \frac{1}{(z+a)(z-b)}$, $|a| < |b|$. [10 pts]

i. Expand f in a Laurent series valid in $|a| < |z| < |b|$.

ii. Use the Laurent series expansion obtained above to calculate

$\int_{|z|=\frac{|a|+|b|}{2}} z^{10} f(z)dz$ and $\frac{d^{10} f(0)}{dz^{10}}$.

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